# Nonlinear Dynamics and Quantum Entanglement in Optomechanical Systems

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To search for and exploit quantum manifestations of classical nonlinear dynamics is one of the most fundamental problems in physics. Using optomechanical systems as a paradigm, we address this problem from the perspective of quantum entanglement. We uncover strong fingerprints in the quantum entanglement of two common types of classical nonlinear dynamical behaviors: periodic oscillations and quasiperiodic motion. There is a transition from the former to the latter as an experimentally adjustable parameter is changed through a critical value. Accompanying this process, except for a small region about the critical value, the degree of quantum entanglement shows a trend of continuous increase. The time evolution of the entanglement measure, e.g., logarithmic negativity, exhibits a strong dependence on the nature of classical nonlinear dynamics, constituting its signature.

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Entanglement [1,2], a form of quantum superposition, is a fundamental phenomenon constituting the cornerstone of quantum computing and quantum information science [3]. The phenomenon has been observed in experiments of large molecules [4] and even small diamonds [5]. Recently, there is also great interest in entanglement in optomechanical systems [6–12].

While the principle of superposition is defined with respect to linear systems, our physical world, when viewed classically, is nonlinear. A nonlinear dynamical system can exhibit all kinds of interesting phenomena such as periodic oscillations, quasiperiodicity, and chaos [13], and they can have distinct fingerprints or manifestations when the system is treated quantum mechanically. Especially, the studies of quantum manifestations of classical chaos constitute the field of quantum chaos, and there have been tremendous efforts in the past four decades [14] in this field. With respect to entanglement, there is no classical correspondence in the strict sense, as the measurement of a physical state at one place immediately influences the measurement at the other. In view of the ubiquity of nonlinear dynamics in physical systems and of the fundamental importance of quantum entanglement, curiosity demands that we ask the following question: What is the interplay between nonlinear dynamics and quantum entanglement? In this regard, earlier works indicated that the transition from integrability to quantum chaos is related to quantum-phase transition in the Dicke model [15], and the transition is accompanied by the emergence of an entanglement singularity in the quantum cusp catastrophe model [16,17]. Signature of nonlinear behavior in entanglement may have significant practical implications in the development of devices and systems for quantum computing and quantum information processing.

In this Letter, we address the nonlinear-dynamics–quantum-entanglement issue by using optomechanical systems, a field of intense recent investigation [18–31]. An optomechanical system consists of an optical cavity and a nanoscale mechanical oscillator, such as a cantilever. When a laser beam is introduced into the cavity, a resonant optical field is established that exerts a radiation force on the mechanical cantilever, causing it to oscillate. The mechanical oscillations in turn modulate the length of the optical cavity, hence, its resonant frequency. There is, thus, coupling between the optical and the mechanical degrees of freedom. This coupling, or interaction, can lead to cooling of the mechanical oscillator toward the quantum ground state, a topic of great interest [32,33]. The optomechanical coupling, thus, provides a straightforward way to entangle the optical with the mechanical modes, and this can have profound implications to optical information science and quantum computing [11,34]. With regard to nonlinear dynamics, there were experimental works reporting chaos in optomechanical systems [31,35,36].

There are recent theoretical works on quantum entanglement in optomechanical systems [6–12]. For example, a protocol was proposed for entanglement swapping with application to optomechanical systems [9], and recent work revealed robust photon entanglement in optomechanical interfaces [10] and ways to achieve strong steady-state entanglement in a three-mode optomechanical system [11]. The interplay between synchronization and entanglement in optomechanical systems has also been explored [12]. However, existing works focused on the situation where the classical dynamics are either steady-state [6,8–12] or periodic oscillations [7]. Our goal is to search for nonlinear dynamical behaviors *beyond* and to investigate the effects of such behaviors on quantum entanglement. Our main findings are the following. In an experimentally realizable parameter regime, as the power of the driving laser is increased, there is a transition from periodic to quasiperiodic motions, where in the latter, the system is strongly nonlinear with two incommensurate frequencies. Entanglement is enhanced towards the transition, vanishes as the transition point is being reached, but is restored abruptly after the transition and continues to be enhanced as the system evolves deeply into the quasiperiodic regime. A surprising result is that, with respect to time evolution, there are direct signatures of classical nonlinear dynamics in quantum entanglement. In particular, for classically periodic dynamics, the time evolution of the entanglement measure is also periodic, but when the classical system enters into the quasiperiodic regime, the quantum entanglement measure exhibits a beats-like behavior with two distinct frequencies. Entanglement, especially when the classical system is quasiperiodic, is robust with respect to temperature variations.

*Model.*—We consider a generic type of optomechanical systems, as shown in Fig. 1(a). Such a system is essentially a Fabry-Perot cavity with a fixed, partially reflecting mirror at the left side and a movable, perfectly reflecting end mirror on the right side. The cavity has equilibrium length $l_0$ and finesse $F$. The movable mirror is attached to a mechanical oscillator of mass $m$, characteristic frequency $\omega_M$, and dissipation rate $\Gamma_M$. The fixed end of the cavity has the optical decay rate $\kappa = \pi c/(2Fl_0)$. We assume that a single optical mode of frequency $\omega_c$ is excited by an external laser field of frequency $\omega_0$, where $\omega_c \sim \omega_0$ so that the detuning is on the order of the mechanical frequency $\omega_M$. The Hamiltonian of the system can be written as [6,37,38] $\hat{H} = \hbar\omega_c \hat{a}^\dagger \hat{a} + \hbar\omega_M \hat{b}^\dagger \hat{b} - \hbar g_0 \hat{a}^\dagger \hat{a}(\hat{b}^\dagger + \hat{b}) + i\hbar(E e^{-i\omega_0 t}\hat{a}^\dagger - E^* e^{i\omega_0 t}\hat{a}) + \hat{H}_\kappa + \hat{H}_{\Gamma_M}$, where $\hat{a}(\hat{a}^\dagger)$ and $\hat{b}(\hat{b}^\dagger)$ are the annihilation and creation operators associated with the optical and mechanical modes, respectively. The first two terms in the Hamiltonian describe the frequencies of the uncoupled optical and mechanical components, respectively, and the third term represents the change in the characteristic frequency of the optical mode as a result of the mirror movement. The quantity $g_0 = G\sqrt{\hbar/(2m\omega_M)}$ is the vacuum optomechanical coupling strength characterizing the interaction between one single photon and one single phonon—see Sec. S1 in the Supplemental Material [39]. The quantity $\sqrt{\hbar/(2m\omega_M)}$ is the mechanical zero-point fluctuation with effective mass $m$, and $G \simeq \omega_c/l_0$. The fourth term describes the interaction between the cavity field and the driving laser field of complex amplitude $E$, where the input laser power is $|E|^2 = 2\kappa P/\hbar\omega_0$. The last two terms in the Hamiltonian represent the dissipation resulting from the cavity decay and mechanical friction, respectively.

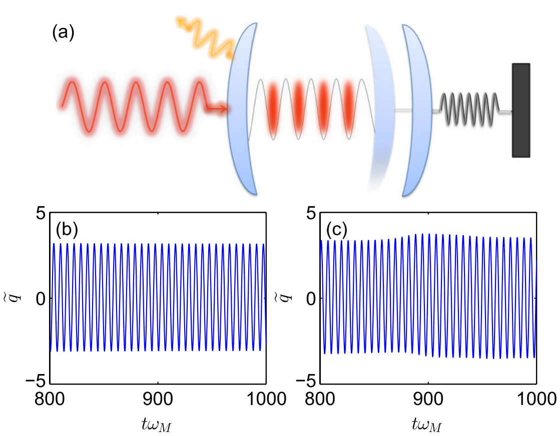


FIG. 1 (color online). (a) Schematic illustration of a typical optomechanical system driven by a single-mode laser. The laser beam enters the optical cavity through the fixed mirror, but photons inside the cavity can also decay through the mirror. The movable mirror is totally reflecting and linked to a frictional mechanical oscillator. [(b),(c)] Time series of the dimensionless canonical coordinate $\tilde{q}$ for the situations where the classical dynamics are periodic and quasiperiodic, respectively.

*Methods.*—To explore and exploit nonlinear behaviors in the optomechanical system, we use the Heisenberg equations of motion [40]. The dynamics of the optical-cavity field can be described by its corresponding complex amplitude, and the motion of the mechanical mode can be characterized by a pair of canonical coordinates $q$ and $p$. By replacing the photon annihilation (creation) operators by the complex light amplitude and the position operator of the cantilever by its classical counterpart in the Heisenberg equations, we obtain the classical equations of motion [40]. This can effectively be regarded as a classical system described by four independent variables, two associated with the optical mode and two with the mechanical mode. In the language of nonlinear dynamics, the optomechanical system in Fig. 1 has a four-dimensional phase space.

The degree of quantum entanglement can be quantitatively assessed by using the corresponding quantum Langevin equations. Following the standard input-output theory [41] and making use of the reference frame rotating at the laser frequency, we can get the following nonlinear quantum Langevin equations [6,7]:

$$\dot{q} = \omega_m p, \quad \dot{p} = -\omega_m q - \Gamma_M p + \sqrt{2} g_0 a^\dagger a + \xi,$$
$$\dot{a} = -(\kappa + i\Delta_0)a + i\sqrt{2} g_0 a q + E + \sqrt{2\kappa}a^{\mathrm{in}}, \tag{1}$$

where $\Delta_0 = \omega_c - \omega_0$ is the cavity detuning. Here, we assume that the optical bath is in thermal equilibrium, i.e., in some incoherent state, so that $a^{\mathrm{in}}$ can be interpreted as a noise term [41]. The vacuum noise-input term can be described by a zero-mean Gaussian random process. The autocorrelation function of the vacuum noise is $\langle a^{\mathrm{in}}(t) a^{\mathrm{in}\dagger}(t')\rangle = \delta(t-t')$. The mechanical mode is under the influence of stochastic Brownian noise that satisfies the non-Markovian autocorrelation relation [6,42,43]: $\langle \xi(t)\xi(t')\rangle = (\Gamma_M/\omega_M)\int (2\pi)^{-1} e^{-i\omega(t-t')}\omega\{\coth[\hbar\omega/(2k_B T)] + 1\}d\omega$.

For the quantum effect of the mechanical mode to be important, the quality factor of the mechanical oscillator must be sufficiently high: $Q = \omega_M/\Gamma_M \gg 1$. In this regime, we have the following Markovian delta-correlated relation: $\langle \xi(t)\xi(t') + \xi(t')\xi(t)\rangle/2 = \Gamma_M(2\bar{n}+1)\delta(t-t')$, where $\bar{n} = 1/(\exp[\hbar\omega_M/(k_B T)] - 1)$ is the mean mechanical phonon number and $T$ is the temperature of the mechanical bath. Equation (1) is a set of quantum stochastic equations that can be simulated numerically (cf., Sec. S3 in the Supplemental Material [39]), and the standard ensemble method can be used to calculate the degree of quantum entanglement. Another method was introduced in Ref. [7], where a time-dependent covariance matrix is calculated to fully characterize the evolution of the Gaussian state. We use both methods and set the system in the weakly coupling regime so that the magnitudes of noise are small compared to the zeroth-order effects (cf., Sec. S2 in the Supplemental Material [39]).

In the Gaussian-evolution method, we assume small fluctuations so that the relevant quantum operators can be expanded about their respective mean values: $O(t) = \langle O\rangle(t) + \delta O(t)$, where $O \equiv (q, p, a, a^\dagger)$, yielding a set of inhomogeneous equations governing the time evolution of the fluctuations $\dot{u}(t) = A(t)u(t) + n(t)$, where $A(t)$ is a $4\times 4$ matrix and $n(t)$ characterizes the input noise (see Sec. S4 in the Supplemental Material [39]). We can also obtain a set of equations for the mean values of the operators, but they have the same forms as the corresponding classical equations of motion. Whereas Eq. (1) is nonlinear, $\dot{u}(t)$ is a set of stochastic linear equations, meaning that the fluctuations will evolve asymptotically into zero-mean Gaussian states but only when none of the Lyapunov exponents in the corresponding classical system is positive. The covariance-matrix approach is, thus, not applicable to situations where the classical dynamics are chaotic [44]. For a nonchaotic dynamical process, the properties of the quantum fluctuations can be determined from the $4\times 4$ covariance matrix that obeys [7,42] $\dot{V}(t) = A(t)V(t) + V(t)A^T(t) + D$, where the element of the covariance matrix is defined as $V_{ij} = \langle u_i u_j + u_j u_i\rangle/2$, and $D = \mathrm{diag}(0, \gamma_M(2\bar{n}+1), \kappa, \kappa)$. This comes from $\langle n_i(t)n_j(t') + n_j(t')n_i(t)\rangle/2 = \delta(t-t')D_{ij}$, characterizing the magnitudes of the noisy terms. For convenience, we can express $V$ as

$$V = \begin{bmatrix} V_A & V_C \\ V_C^T & V_B \end{bmatrix},$$

where $V_A$, $V_B$, and $V_C$ are $2\times 2$ matrices associated with the mechanical mode, the optical mode, and the optomechanical correlation, respectively. The degree of quantum entanglement between the mechanical and optical modes can, thus, be assessed by calculating the so-called *logarithmic negativity*, defined as [45] $E_N \equiv \max[0, -\ln(2\eta^-)]$, where $\eta^- \equiv (1/\sqrt{2})[\Sigma(V) - \sqrt{\Sigma(V)^2 - 4\det V}]^{1/2}$ and $\Sigma(V) = \det(V_A) + \det(V_B) - 2\det(V_C)$ (cf., Sec. S5 in the Supplemental Material [39]). We define $E_p \equiv -\ln(2\eta^-)$ as the pseudoentanglement measure so that $E_N = \max(0, E_p)$.

For the ensemble method, we simulate Eq. (1) a large number of times to obtain time series of the fluctuations. We can then obtain the time series of the covariance matrix from its definition, i.e., $V_{ij} = \langle u_i u_j + u_j u_i\rangle/2$. The entanglement degree can be calculated from the definition of $E_N$.

*Results.*—We find that there are wide parameter regimes in which the optomechanical system exhibits periodic and quasiperiodic motions, the representative time series of which are shown in Figs. 1(b) and 1(c), respectively. Whereas quasiperiodic motions and even chaos have been discovered recently in a coupled BEC (Bose-Einstein condensation) type of hybrid optomechanical system [46], to our knowledge there were no previous reports of quasiperiodic motions in the generic optomechanical systems as in Fig. 1, which typically occur for low values of the cavity-decay rate, e.g., $\kappa < 0.2\omega_M$, and for moderate values of the detuning, e.g., for $\Delta_0 \approx -0.8\omega_M$. The dissipation rate of the mechanical oscillator can take on values from a large range, e.g., $10^{-5}\omega_M \sim 10^{-3}\omega_M$. For computational convenience, we rescale the dynamical variables: $\tilde{q} = \sqrt{2}g_0 q/\omega_M$ and $\tilde{P} = 8g_0^2E^2/\omega_M^4$. Using the classical Heisenberg equations, we calculate the bifurcation diagram of the system, i.e., the asymptotic extreme values of the dynamical variables as a function of the power of the driving laser, as exemplified in Fig. 2(a), for the experimentally reasonable parameter setting of $\omega_M = 2\pi$MHz, $Q = 25000$, and $m = 10.67$ ng. For this diagram, the cavity is assumed to be driven by a blue detuned laser with value of the detuning $\Delta_0 = -0.81\omega_M$ and wavelength $\lambda = 1064$ nm. The damping of the mechanical mode is related to the quality factor $Q$ by $\Gamma_M = \omega_M/Q$. To ensure the validity of the photon-pressure Hamiltonian, we estimate $x/l_0 = \tilde{q}\omega_M/\omega_c \sim 10^{-8}$, so the small-displacement assumption is well justified [37,40]. As the normalized laser power is increased through a threshold value $\tilde{P}_c$, a transition from periodic oscillation to quasiperiodic motion occurs. Figure 2(b) shows the Lyapunov spectrum versus $\tilde{P}$. We observe that for $\tilde{P} < \tilde{P}_c$, the maximum Lyapunov exponent is zero but there is only one zero exponent, signifying the existence of a periodic attractor. For $\tilde{P} > \tilde{P}_c$, the maximum Lyapunov exponent remains to be zero but there are two such exponents. In particular, the second largest Lyapunov exponent is negative for $\tilde{P} < \tilde{P}_c$ but it becomes zero for $\tilde{P} > \tilde{P}_c$, a feature characteristic of the transition from periodicity to quasiperiodicity. In the range of $\tilde{P}$ values shown, the third and the fourth Lyapunov exponents are all negative. The feature that there is no positive Lyapunov exponent renders applicable the use of the quantum Langevin equation to calculate the degree of quantum entanglement, because zero-mean Gaussian random inputs lead to fluctuation patterns that remain Gaussian. This preservation of the Gaussian distribution means that the fluctuations can be fully determined by the covariance matrix. Figure 2(c) shows the pseudo-entanglement maximum $E_{p,m}$ as a function of $\tilde{P}$. Note that there is entanglement when the value of $E_p$ is greater than 0, so

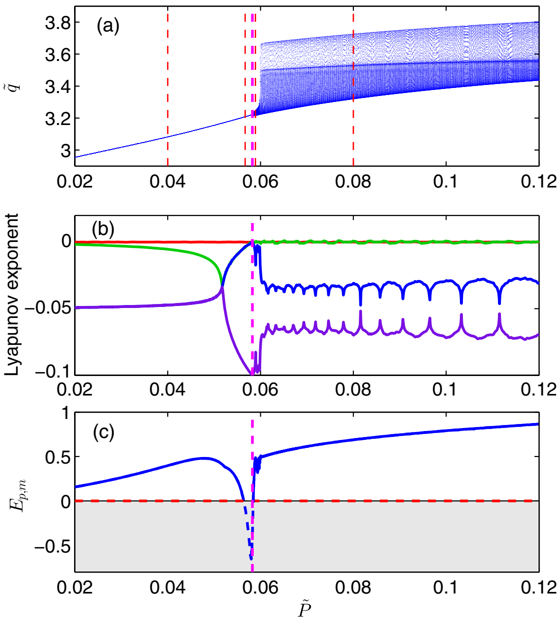


FIG. 2 (color online). (a) Typical bifurcation diagram of the generic optomechanical system in Fig. 1(a), where the asymptotic extreme values of the dimensionless canonical coordinate $\tilde{q}$ are plotted versus the normalized external laser power $\tilde{P}$. The thick purple dashed line indicates the threshold driving power $\tilde{P}_c = 0.05842$ at which a transition from periodic to quasiperiodic motions occur. The cavity length is $l_0 = 25$ mm with finesse $F = 6 \times 10^4$. (b) The entire spectrum of the four Lyapunov exponents versus $\tilde{P}$ and (c) the maximum of the pseudoentanglement measure $\bar{E}_p$ versus $\tilde{P}$. The thin red vertical dashed lines at $\tilde{P} = 0.0400$, 0.0567, 0.0589, and 0.0800 in panel (a) indicate the specific values of $\tilde{P}$ at which the time evolution of $E_p$ is to be examined (see Fig. 3).

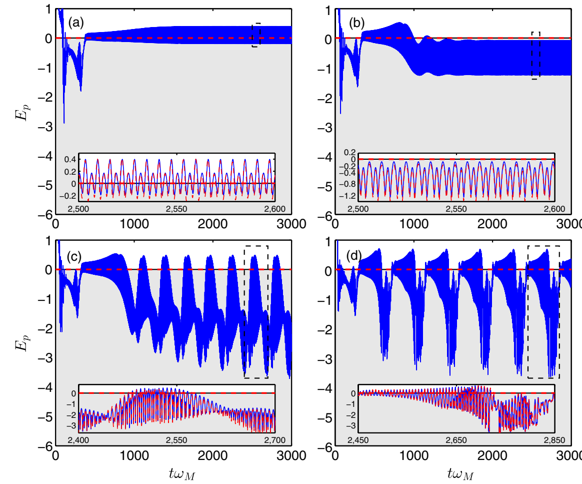


FIG. 3 (color online). (a)–(d) Time evolution of $E_p$ for four values of the laser driving power $\tilde{P}$ as specified by the vertical red dotted lines in Fig. 2(a). Insets show the respective zoom-in regions indicated by the black dashed boxes, and the thin red dashed lines represent the results from the ensemble approach. The shaded regions correspond to nonphysical results and $E_N = \max[0, E_p]$.

the shaded region in which the values of $E_{p,m}$ fall below zero indicates lack of entanglement. The surprising phenomenon is that, as the system evolves towards the transition point, the value of $E_{p,m}$ continues to increase but drops to zero rather abruptly as the transition is reached. In the vicinity of the transition, quantum entanglement disappears. As the classical dynamics becomes quasiperiodic, entanglement is restored immediately because the value of $E_{p,m}$ is recovered and continues to increase as the laser power is further increased. The basic observation is that strongly nonlinear behavior can lead to the enhancement of quantum entanglement between the optical and mechanical modes.

To probe further into the dynamics of quantum entanglement through the classical transition point, we calculate the time evolution of the pseudoentanglement measure $E_p(t)$ for representative values of the driving laser power, as shown in Fig. 3 for four cases. Results from the ensemble approach by simulating the quantum Langevin equations with 3000 realizations are also included. We observe a strong correlation between $E_p(t)$ and the evolution of the classical dynamics, in that $E_p(t)$, after a short transient period, exhibits periodic (quasiperiodic) behavior if the classical dynamics is periodic (quasiperiodic). In all cases of $E_p > 0$ so that there is entanglement, the phenomenon of death and rebirth of entanglement [47,48] with time occurs, in which $E_p(t)$ becomes negative and then restores to some positive value after a time interval. From the perspective of applications, the entanglement duration time is of the order of a microsecond so that the effects of entanglement oscillations are negligible when the desirable quantum operations can be achieved at sufficiently high speed. The remarkable phenomenon is that when quasiperiodicity sets in so that the classical dynamics possesses two incommensurate frequencies, the corresponding quantum pseudoentanglement dynamics, after its "rebirth," exhibits a surprising "beats" or temporal modulation phenomenon. There is, thus, a direct consequence of classical nonlinear characteristics in quantum entanglement.

We remark that there are proposals to quantify the entanglement experimentally [6,49,50]. In particular, with the aid of an ancillary cavity, one can construct the covariance matrix through homodyne detection and then calculate the logarithmic negativity.

To summarize, using optomechanical systems as a paradigm, we have addressed the manifestations of classical nonlinear dynamics in quantum entanglement, with a focus on two common types of classical dynamics: periodic and quasiperiodic motions. Our result is that strong signatures of the classical dynamics exist in the respective quantum entanglement dynamics. For example, when the classical dynamics is quasiperiodic, the corresponding quantum entanglement exhibits a surprising "beating" behavior in its time evolution. Not only is the degree of entanglement enhanced as the classical dynamics transits from periodic to quasiperiodic motions but the entanglement corresponding to the latter is also more temperature robust (cf., Sec. S6 in the Supplemental Material [39]). Pushing the classical system into a highly nonlinear regime so that it exhibits more complicated motion than periodic oscillation can, thus, be beneficial for achieving quantum entanglement [44].

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to calculate the logarithmic negativity based on the evolution of the covariance matrix is not applicable. New methodology needs to be developed to characterize quantum entanglement for classically chaotic systems. In this regard, the relation between entanglement and phase transition in the Dicke model has been investigated [16], where the role of phase transitions in the connection between entanglement and underlying integrable to quantum chaotic transitions was speculated. The quantum-chaotic properties of the Dicke Hamiltonian were also explored [15], and entanglement in cusp quantum catastrophes was examined [17].
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