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Hollowing strategies for enhancing robustness of geographical networks

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Abstract. – Geographical networks have attracted much attention recently, and it has been shown that they are more fragile under node failures than those without geographical constraints, due to the large amount of small-order cycles in these networks. In this paper, a hollowing strategy is proposed to break the small-order cycles in geographical networks and make them form higher-order cycles, thus making the networks more robust, at the same time keeping the cost low. The validity of the strategy is investigated both analytically and numerically. The hollowing strategy could be applicable in real geographical systems in times of crisis.

Complex networks (see reviews [1–4]) provide a powerful tool to investigate complex systems in nature and society. In many real complex systems, the properties of networks are affected by the geographical distribution of nodes and edges; for example, the routers of Internet and social networks lie on the two-dimensional surface of the globe, neural networks in brains occupy three-dimensional space, etc. The transmission lines of power grid networks or the cables in Internet router networks are nearly always connecting the nearest spatial neighbors in order to reduce the cost. Thus, it is natural to study the geographical complex networks and this has attracted much attention recently [5–17].

It has been shown that the geographical structure has crucial influence on percolation thresholds, which is defined as the point where the differential of the size of the largest cluster as a function of the occupying probability q peaks [18]. For scale-free networks, *i.e.*, the degree distribution $P(k)$, the probability that a node has k connections, follows a power law form $P(k) \sim k^{-\lambda}$; if the nodes are randomly connected, when $\lambda \leqslant 3$, the network has 0 percolation threshold for random node failures in the large network size limits [19, 20], that is, no matter how small the fraction of nodes that remained active is, there will exist a giant cluster connecting them. But if the same network is embedded in a 2D lattice with nearest-neighbor connections, the network will have a finite percolation threshold [5], due to the large amount of small-order cycles in the network. Also, many authors have demonstrated that when a network is less geographically constrained (more loosely connected from the geographical point of view), and has less small-order cycles in the abstract topology aspect, it will be easier to percolate, and more robust under random node failures and intentional attacks [21].

In [20, 22] a generating function process is applied to the percolation problem of random networks (limited to tree-like networks for large network size), and the authors obtain the percolation threshold q_c as the condition that the average cluster size diverges, or equivalently, the average size of clusters that are reached by following an edge diverges. For uniform occupations (or random node failures), the percolation threshold is known as $q_c = \langle k\rangle/\langle k(k-1)\rangle$ [19,20]. To comprehend the phenomena that more loosely connected geographical networks (with less small-order cycles) percolate more easily, in [21] the authors extended the result of percolation threshold from tree-like networks to networks containing triangles, considering the effects of redundant edges in triangles which have no contribution in forming larger clusters. Similar considerations also appear in [23], which considers the effects of clustering on the formation of the giant component in a random network.

Based on the numerical observation and analysis in [21] that geographically more loosely connected networks (with less small-order cycles) are more robust, in this paper, we propose a hollowing strategy to enhance the robustness of geographical networks, with possibly minimal cost. The validity of the strategy is investigated both analytically and numerically, *i.e.*, by analysing the effects of arbitrary order cycles on the percolation threshold in clustered random graphs with arbitrary degree distributions and by numerical simulations of the percolation threshold upon various hollowing grades in several network cases. Particularly, our analytical results illustrate that cycles in networks help form heavily inter-connected clusters, which makes the network discrete and fragile under random node failures or intentional attacks, and increases the percolation thresholds; furthermore, higher-order cycles (or large- range edges) have much diminished influence on percolation thresholds, which might provide fundamental insights into the stability of many real systems. Although the analysis is on clustered random graphs, it could provide a broad understanding of the effects of cycles on percolation problems of natural and artificial systems.

The starting point of our hollowing strategy is: For a normal geographical network with arbitrary degree distribution, restrained by the cost of construction, its connections are usually settled between nodes and their spatially nearest neighbors, which makes the network locally clustered. In an abstract topological view, the network has a large amount of small-order cycles, which makes the network form small tightly connected and fractionalized clusters. However, if the edges in small-order cycles are broken and form new larger-order cycles, the network will be less localized and the edges will be distributed more homogeneously in the geographical view, and be more efficiently used, which make the network more robust under network accidents, such as random node failures or intentional attacks. Thus our process is, for each node in a given geographical network, it has probability p to cut down the edges that linked to its first n nearest neighbors, then reconnects them also to nearest neighbors excluding the first n nodes, if they have not been connected before [24, 25]. The reconnection is cost effective, *i.e.*, beside increasing robustness, it has minimal cost increase. In this case, only the *cost* of physical connections is accounted for; the cost of reconstructing nodes and other level of costs [26] are neglected. Longer connections (with more cost) will be more effective, as that in the Watts and Strogatz's small-world model [18], but as we will see, this minimum hollowing strategy already shows high efficiency. It should be noted that the hollowing process is not dependent on lattice-embedded network, but only touches upon geography, *i.e.*, it only considers the spatial distance between the nodes in a geographical network, and is valid regardless of how the network is constructed, whether lattice-embedded [5–12] or through spatial length-dependent growth [13–16]. In the following simulations, we mainly focus on hollowing a lattice-embedded scale-free (LESF) model [6, 7], generated as follows: Each node on an $L \times L$ lattice with periodic boundaries is assigned a degree k, drawn from a scale-free degree distribution: $P(k) \sim k^{-\lambda}$, $k \geqslant m$, where m is the minimum degree of the network.

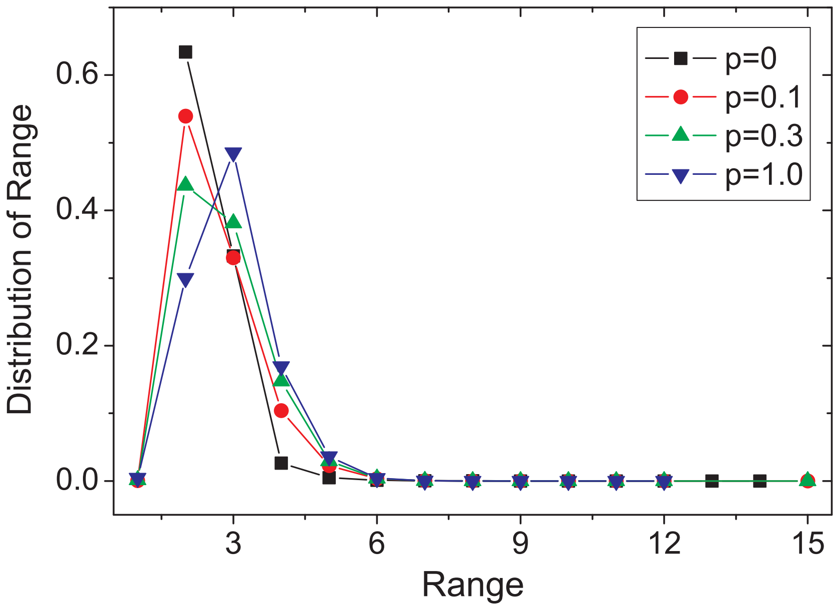

Fig. 1 – Range distribution of an LESF network under certain hollowing grades. Points with range $= \infty$ are plotted at range $= 1$. The degree distribution exponent is $\lambda = 5.0$, the hollowing parameter $n = 8$, the minimum degree $m = 4$, and the network size $N = 10^6$. Each datum (cf. fig. 2) is obtained from one realization.

Then a node is picked out at random and connects to its nearest neighbors, until its previously assigned degree k is realized, or until all nodes are up to a distance $Ak^{1/d}$ with the lattice dimension $d = 2$. This process is repeated for all nodes of the lattice. Generally, A is large enough ($A = 7$ in this paper) to make the assigned degrees be almost realized.

First, we will illustrate that the hollowing process breaks the small-order cycles of the geographical networks, employing the concept of "range". The range of an edge is the length of the shortest path between the nodes it connected in the absence of itself [27–29]. If an edge's range is l, the shortest cycle it lies on is of length $l+1$. Thus the distribution of range sketches the distribution of cycles of various orders. Figure 1 demonstrates the range distribution of an LESF network under various hollowing grades. It verifies that as the geographical network is more hollowed, the network has more large-range edges and less small-range edges. Thus in the hollowed network, small-order cycles are broken and large-order cycles are formed, and the locality of connections is lower.

Next, we will demonstrate that for random networks with arbitrary degree distribution $P(k)$ the higher-order cycles have much diminished influence than the smaller-order cycles on increasing percolation thresholds. To assess the influence by higher-order cycles, we begin by considering the influence of L-cycles separately as follows, under the assumption that the cycles are distributed sparsely in the network, *i.e.*, each node belongs to one cycle at most, and in average, $n_L(k) < 1$ or ~ 1.

For a uniform occupation probability q, the generating functions for the probability of the number of outgoing edges by following a randomly chosen edge on a clustered network remain the same as that of the tree-like networks [20, 22]:

$$F_1(x) = \frac{q}{\langle k \rangle} \sum kP(k)x^{k-1}. \tag{1}$$

But if an outgoing edge is known to be not independent, or terminated, *i.e.*, it connects to an existing node of the local cluster, thus the node with degree k has only $k-2$ independent edges, reaching out $k-2$ new nodes. Thus it is convenient to define

$$F_1^{(1)}(x) = \frac{q}{\langle k \rangle} \sum kP(k)x^{k-2} = x^{-1}F_1(x), \tag{2}$$

as the generating function of the number of outgoing edges reaching new nodes by following an edge that has such a target node.

Let $H_1(x)$ be the generating function of the size distribution of the cluster that is reached by following an edge; by following 2 independent edges, the cluster size distribution is generated by $H_1(x)^2$. But if the 2 edges originate from a common node and belong to an L-cycle, the generating function should be

$$H_1^{(m_1)}(x)H_1^{(m_2)}(x),$$

where $m_1 = [(L-1)/2]$, $m_2 = L-1-m_1$, $[x]$ is Gauss' function, which yields the largest integer that is less than or equal to x, and $H_1^{(m)}(x)$ is the generating function for the size distribution of the clusters that are reached by an edge and have one edge terminated after m steps, and satisfies the iterative relation

$$H_1^{(m)}(x) = 1 - F_1(1) + xF_1^{(1)}(H_1(x))H_1^{(m-1)}(x),$$

where the terminal condition $H_1^{(1)}(x)$ is

$$H_1^{(1)}(x) = 1 - F_1(1) + xF_1^{(1)}(H_1(x)).$$

Thus

$$H_1^{(m)}(x) = (1-F_1(1))\frac{1-(xF_1^{(1)}(H_1(x)))^m}{1-xF_1^{(1)}(H_1(x))} + (xF_1^{(1)}(H_1(x)))^m.$$

In general, we may assume that a node i with degree k_i reached by following an edge belongs to $n_L(k)$ L-cycles, thus $H_1(x)$ satisfies the self-consistent equation

$$H_1(x) = 1 - F_1(1) + \frac{qx}{\langle k \rangle} \sum kP(k)[H_1^{(m_1)}(x)H_1^{(m_2)}(x)]^{n_L(k)} H_1(x)^{k-1-2n_L(k)}.$$

The average cluster size reached by an edge is

$$\begin{aligned}\langle \widetilde{s} \rangle &= H_1'(1)\\ &= q + \frac{q}{\langle k \rangle} \sum kP(k)[n_L(k)(H_1^{(m_1)\prime}(1) + H_1^{(m_2)\prime}(1)) + (k-1-2n_L(k))H_1'(1)],\end{aligned}$$

where

$$H_1^{(m)\prime}(1) = \frac{1-q^m}{1-q}(xF_1^{(1)}(H_1))'(1) = \frac{1-q^m}{1-q}H_1^{(1)\prime}(1),$$

and

$$H_1^{(1)\prime}(1) = q + q\frac{\langle k(k-2) \rangle}{\langle k \rangle}H_1'(1).$$

A simple substitution yields

$$\langle \widetilde{s} \rangle = \frac{q + \frac{2-q^{m_1}-q^{m_2}}{1-q}\frac{q^2}{\langle k \rangle}\langle kn_L(k) \rangle}{1 - \frac{q}{\langle k \rangle}\left[\langle k(k-1) \rangle - 2\langle kn_L(k) \rangle\left(1 - \frac{2-q^{m_1}-q^{m_2}}{2(1-q)}\frac{q\langle k(k-2) \rangle}{\langle k \rangle}\right)\right]}.$$

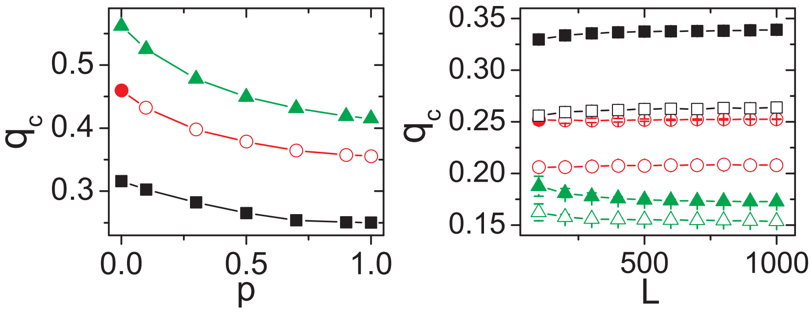

Fig. 2 – Random percolation thresholds q_c of LESF networks before and after hollowing. Left panel: q_c *vs.* the rearranging probability p: squares for $\lambda = 3.0$, circles for $\lambda = 5.0$, and triangles for $\lambda = 10.0$; the hollowing parameter is $n = 8$, the network size $N = 10^6$, and each series was obtained averaging over 1000 replicas. Right panel: q_c *vs.* network side length L for the LESF model before (filled symbols) and after (empty symbols) the hollowing process with $n = 8$ and $p = 1$ for different λ: squares for $\lambda = 2.8$, circles for $\lambda = 2.5$, and triangles for $\lambda = 2.3$; each datum was obtained by averaging over 10^4 replicas. For both panels, the minimum degree is $m = 4$. Note that the error bars are usually smaller than the symbols.

Thus the percolation threshold q_c, given by the divergence of the average size of clusters that is reached by an edge, is

$$q_c = \frac{\langle k\rangle}{\langle k(k-1)\rangle - 2\langle kn_L(k)\rangle\left(1 - \frac{2-q_c^{m_1}-q_c^{m_2}}{2(1-q_c)}\frac{q_c\langle k(k-2)\rangle}{\langle k\rangle}\right)}. \qquad (3)$$

This result degenerates to the known result of tree-like networks $\langle k\rangle/\langle k(k-1)\rangle$ when $n_L = 0$. For 3-order cycles, with $L = 3$, $m_1 = m_2 = 1$, and $n_3(k) = C(k)(k-1)^2/2$. Equation (3) reduces to [21]

$$q_c = \frac{\langle k\rangle}{\langle k(k-1)\rangle - (1 - q_c\frac{\langle k(k-2)\rangle}{\langle k\rangle})\langle C(k)k(k-1)^2\rangle}. \qquad (4)$$

For another limiting case, consider the infinite-order cycles: As $L \to \infty$, $q_c^{m_1}$ and $q_c^{m_2}$ tends to zero, eq. (3) reduces to a 2-order equation for q_c, and the meaningful root is $q_c = \langle k\rangle/\langle k(k-1)\rangle$, which is just the percolation threshold of tree-like networks. This means that cycles of infinite length have no effect on percolation thresholds. In general, for identical number of cycles, the higher the cycle order L is, the smaller $q_c^{m_1} + q_c^{m_2}$ will be, thus the less influence it will have. Numerical tests for some typical data values also show that q_c is an increasing function of $q_c^{m_1} + q_c^{m_2}$. In the above analysis the degree distribution could have arbitrary forms, so the results are valid independent of the degree distributions.

Numerically, we studied the percolation thresholds of the LESF networks before and after hollowing and some other properties. Figure 2 shows a clear drop of the percolation threshold when the network is hollowed, which means that when a network is more heavily hollowed, it will need less nodes to form a giant cluster to exhibit the global functions; thus it illustrates the efficiency of the hollowing strategy for enhancing the robustness of lattice-embedded scale-free networks. The same effect of hollowing on geographical networks with exponential degree distributions has also been observed. Furthermore, the hollowing process can be controlled according to crisis of various extent: light hollow for small crisis, heavy hollow for serious crisis.

After hollowing, the degree distribution deviates a little for small k, *i.e.*, around about 10, as fig. 3 demonstrates. The deviation causes small variations in the moments of degree, which

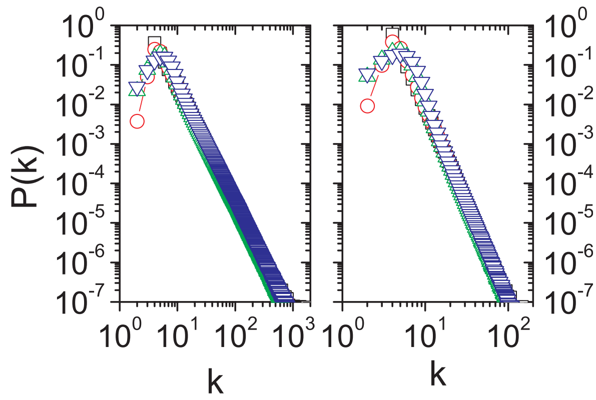

Fig. 3 – Degree distribution of hollowed networks. Left panel: $\lambda = 3.0$; right panel: $\lambda = 5.0$. The hollowing parameter is $n = 8$, and squares: $p = 0$; circles: $p = 0.1$; up triangles: $p = 0.5$; down triangles: $p = 1$. The minimum degree is $m = 4$, the network size $N = 10^6$. Each datum is carried out over 1000 ensembles.

might induce the drop of the percolation thresholds [19, 20]. However, numerical results show that in the $\lambda = 3$ case, the variations in the moments of degree cause no distinct changes in percolation thresholds; in the $\lambda = 5$ case, the variations in degree distribution do cause decreases of percolation thresholds as the rearranging probability p increases, but at a much smaller degree.

In degree-correlated random networks, degree correlation has crucial influence on percolation thresholds [30, 31]. But in the cases studied in the paper, since the degrees of nodes are assigned independently of the geography, while the LESF network are generated with only geographical restrictions, and the reconnection of the hollowing process is regardless of the degree, the degree correlation of these networks tends to zero in the large network size limit, and thus has little influence on percolation thresholds.

The hollowing process has its own real presentations. For example, let us consider the traffic flow network in a city, in which nodes are positions in the city and edges are the real routes that cars travel along: beside the in-city roads, most cities have suburb highways. Under normal conditions, when the traffic is light, the cars travel along the in-city roads since they are short, to reduce costs; suburb highways are almost not used. At this time, the traffic flow network is the nearest-neighbor network. Under the jamming condition, when traffic is heavy, especially when there are jams in the in-city roads, cars could not get through from one position to another, they have to travel through the suburb highways (the hollowing process); thus the traffic flow is mainly distributed on the suburb highways, and little on the in-city roads, which is just the network after a certain hollowness. It should be noted that in this case, "hollowing" has nothing to do with the physical aspects of the urban roads; the "hollowing" process is just the redistribution of the traffic flows, from in-city roads to suburb highways, and the increased cost is the cost of the additional and unwanted gasoline. There are many other real examples, such as the data transmission flow in the Internet when the network graduates from leisure into busy, etc.

In conclusion, a hollowing strategy is proposed to break the small-order cycles and make them form higher-order cycles in geographical networks, thus making the network more stable, at the same time keeping the cost low. The validity of the strategy is investigated both analytically and numerically. Our analytical results illustrate that higher-order cycles (larger-range edges) have much diminished influence on increasing percolation thresholds, which might provide insights into the stability of many real systems. The hollowing strategy could be efficient for real geographical systems in times of crisis, and real applications are also discussed.

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