Supporting Information for

Chiral scars in chaotic Dirac fermion systems

1 Analytic solution of the Dirac equation in circular domain

The Hamiltonian for a massless spin-half particle in a finite domain D in the plane $\boldsymbol{r} = (x, y)$ is given by

$$\hat{H} = -i\hbar v \hat{\sigma} \cdot \nabla + V(\boldsymbol{r})\hat{\sigma}_z, \tag{1}$$

where $\hat{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y)$ and $\hat{\sigma}_z$ are Pauli matrices, and $V(\boldsymbol{r})$ is the infinite-mass confinement potential:

$$V(\boldsymbol{r}) = \begin{cases} 0 & \boldsymbol{r} \in D, \\ \infty & \boldsymbol{r} \text{ outside of } D \end{cases} \tag{2}$$

In the polar coordinates $\boldsymbol{r} = (r, \phi)$ for closed circular domain with radium r_0, the Dirac equation is

$$-i \begin{pmatrix} 0 & \exp(-i\phi)(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \phi}) \\ \exp(i\phi)(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \phi}) & 0 \end{pmatrix} \psi(\boldsymbol{r}) = \mu \psi(\boldsymbol{r}), \tag{3}$$

where $\mu \equiv E/\hbar c$. For a circular boundary, we have $[\hat{J}_z, \hat{H}] = 0$, where $\hat{J}_z = -i\partial_\phi + 1/2\hat{\sigma}_z$ is the total angular-momentum operator. In general, the solutions of Eq. (3) can be expressed as [1]

$$\psi(\boldsymbol{r}) = \begin{pmatrix} C_1 Z_l(\mu r)\exp(il\phi) \\ C_2 Z_{l+1}(\mu r)\exp(i(l+1)\phi) \end{pmatrix}, \tag{4}$$

where $l = 0, \pm 1, \pm 2, ...$, $\mu > 0$ and Z_l is a Bessel function. Substituting Eq. (4) into Eq. (3) and making use of the recursion relations of Bessel functions: $Z_l'(x) \pm l Z_l(x)/x = \pm Z_{l\mp 1}(x)$, we obtain

$$\frac{C_2}{C_1} = i. \tag{5}$$

Since a physical solution must be finite at the origin, we have

$$\psi_{lm}(r, \phi) = N_{lm}\exp(il\phi) \begin{pmatrix} J_l(\mu_{lm} r) \\ i\exp(i\phi)J_{l+1}(\mu_{lm} r) \end{pmatrix}, \tag{6}$$

where the normalization factor

$$N_{lm} = \frac{1}{\sqrt{2\pi \int_0^1 \mathrm{d}r r \left[J_l^2(\mu_{lm} r) + J_{l+1}^2(\mu_{lm} r)\right]}}, \tag{7}$$

and without loss of generality, we have assumed that the radius of the circular domain is unity: $r_0 = 1$, and the eigenvalues μ_{lm} can be determined by the boundary condition: $J_l(\mu_{lm}) = J_{l+1}(\mu_{lm})$. The eigenfunctions satisfy

$$\iint_D \mathrm{d}\boldsymbol{r} \psi_{l'm'}^\dagger(\boldsymbol{r})\psi_{lm}(\boldsymbol{r}) = \delta_{ll'}\delta_{mm'},$$

where the set $\{\psi_{lm}\}$, under the zero-flux boundary condition, forms an orthonormal complete basis for the operator $-i\hat{\sigma} \cdot \nabla$ and its positive integral power $[-i\hat{\sigma} \cdot \nabla]^n$, where $n = 1, 2, \cdots$.

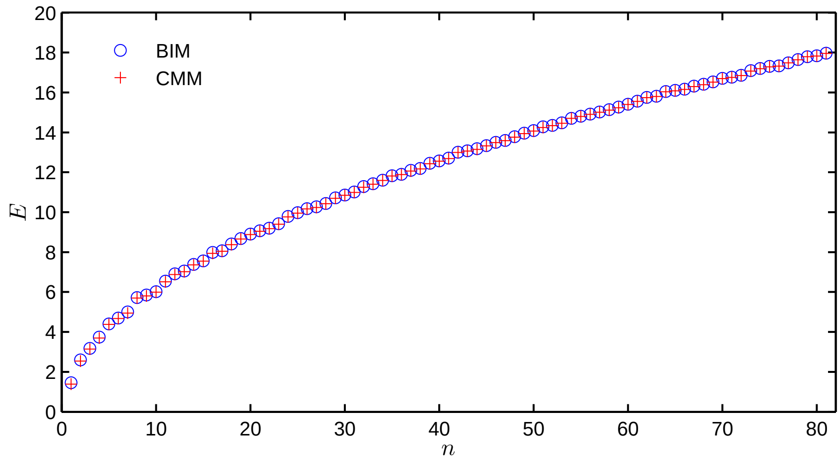

Figure 1: (Color online) Comparison of eigenenergies calculated by using our conformal-mapping method (CMM) and the boundary-integral method (BIM) for a cardioid-shaped Dirac billiard whose classical dynamics is chaotic. Show are the lowest 81 positive eigenenergy levels.

2 Method validation: universal level-spacing statistics in chaotic billiard

Figure 1 shows, for a chaotic billiard ($\beta = 0.49$), the lowest 81 energy levels calculated by two methods: our conformal-mapping based method (CMM) and the boundary-integral method (BIM) due to Berry and Mondragon [2]. We observe a remarkably excellent agreement.

To further validate our method, we calculate the level-spacing statistics, which are believed to exhibit universal behaviors for quantum systems, non-relativistic [3, 4] or relativistic [5], which exhibit chaos in the classical limit. In particular, let $\{k_n | n = 0, 1, 2, \cdots\}$ denote the non-decreasing positive wave-number sequence of a quantum billiard system. According to the Weyl formula [3, 4], the smoothed wave-vector staircase function is given by

$$\langle \mathcal{N}(k) \rangle = \frac{\mathcal{A}k^2}{4\pi} + \gamma \frac{\mathcal{L}k}{4\pi} + \cdots, \tag{8}$$

where $\mathcal{A}$ and $\mathcal{L}$ are the area and perimeter of the billiard, respectively, $\gamma = -1$ (or 1) for Dirichlet (or Neumann) boundary conditions, and $\gamma = 0$ for massless Dirac fermion billiards [2]. Define $x_n \equiv \langle \mathcal{N}(k_n) \rangle$ as the unfolded spectra, which is scaled in units of the mean-level spacing. Let $S_n = x_{n+1} - x_n$ be the nearest-neighbor spacing and $P(S)$ be the probability distribution of S [i.e., $P(S)dS$ is the probability that a spacing S lies between S and $S + dS$]. In the quantum-chaos literature, it has been known [6, 3, 4] that, for classically integrable systems, the level spacing distribution is Poisson: $P(S) \sim \exp(-S)$. For classically chaotic systems that possess time-reversal symmetry but no geometric symmetry, the level-spacing distributions follow the GOE (Gaussian orthogonal ensemble) statistics: $P(S) = (\pi/2)S\exp(-\pi S^2/4)$. For chaotic systems without time-reversal symmetry, $P(S)$ obeys the GUE (Gaussian unitary ensemble) statistics: $P(S) = (32/\pi)S^2\exp(-4S^2/\pi)$. Given $P(S)$, the corresponding cumulative level-spacing distribution can be obtained from $I(S) = \int_0^S \mathrm{d}S' P(S')$. Different types of level-spacing statistics can also be distinguished by using the Δ_3 statistic [3, 4], which is used to measure long-range spectral fluctuations and is defined as

$$\Delta_3(L) = \left\langle \min(a,b) L^{-1} \int_{-L/2}^{L/2} \mathrm{d}x \{N(x_0 + x) - ax - b\}^2 \right\rangle, \tag{9}$$

where the average is over x_0.

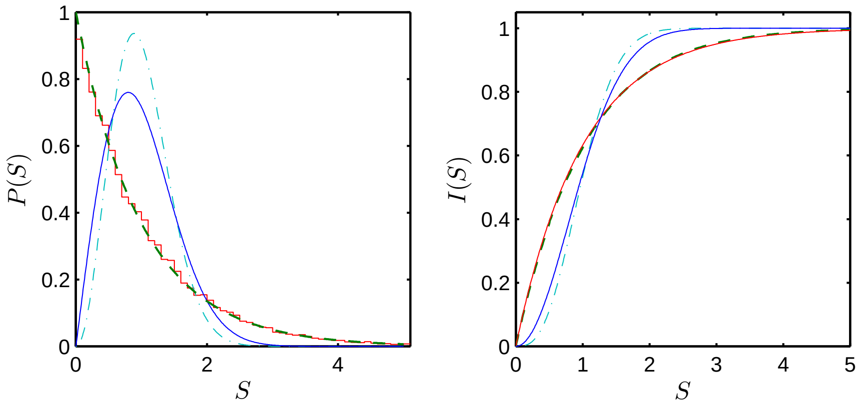

Figure 2: (Color online) Level-spacing statistics for an integrable (circular) Dirac billiard ($\beta = 0$), where the left and right panels are the unfolded level-spacing distribution $P(S)$ and the cumulative level-spacing distribution $I(S)$, respectively. In both panels, numerical data are represented by red line and theoretical distribution curves for Poisson, GOE, and GUE statistics are denoted by the green dash-dotted, blue solid, and cyan dashed curves, respectively.

Utilizing our conformal-mapping method, we are able to calculate a large number of energy levels (on the order of 10^4) for Dirac billiards. Figures 2 and 3 show the statistics of 15,000 energy levels for an integrable ($\beta = 0$) and a chaotic ($\beta = 0.49$) billiard, respectively. We see from the various quantities plotted that the statistics is Poisson for the integrable case but GOE for the chaotic billiard. Note that for the Dirac billiard, the time reversal symmetry is broken, thus one may expect GUE statistics instead of GOE for the chaotic case. However, since the shape of our chaotic billiard has a reflection symmetry, the combination of time-reversal and reflection is preserved, leading to GOE. If the shape has no geometric symmetry, then the resulting level spacing distribution will be GUE. This has been validated numerically by examining level spacing statistics using our method for the Africa billiard employed in Ref. [2].

We emphasize that accurate calculation of such large number of energy levels is unprecedented because no previously known method was able to achieve that.

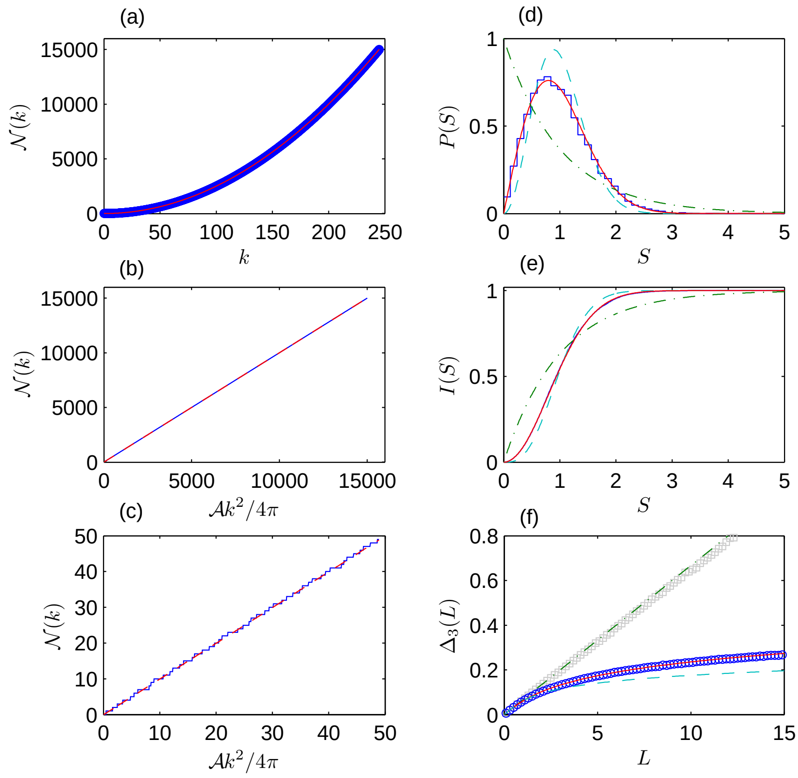

Figure 3: (Color online) Level-spacing statistics for the cardioid-shaped chaotic Dirac billiard ($\beta = 0.49$). (a) Wavevector staircase function $\mathcal{N}(k)$ for the lowest 15,000 levels (circles) versus the wavenumber k. The red curve through the circles is $\langle \mathcal{N}(k) \rangle = \mathcal{A}k^2/(4\pi) - 1/12$. (b) Staircase function $\mathcal{N}(k)$ as a function of $\mathcal{A}k^2/(4\pi)$ (solid curve). (c) Magnification of part of (b) containing the lowest 49 levels. (d) Unfolded level-spacing distribution $P(S)$. (e) Cumulative level-spacing distribution $I(S)$. (f) Spectral rigidity $\Delta_3(L)$. In (d)-(f), green dash-dotted, red solid, and cyan dashed lines denote theoretical distribution curves for Poisson, GOE, and GUE statistics, respectively.

References

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