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Topical Review | Editor's Suggestion

Perspectives on relativistic quantum chaos

Liang Huang[1] **and Ying-Cheng Lai**[2,3]

[1] School of Physical Science and Technology, and Key Laboratory for Magnetism and Magnetic Materials of MOE, Lanzhou University, Lanzhou, Gansu 730000, China
[2] School of Electrical, Computer, and Energy Engineering, Arizona State University, Tempe, AZ 85287, United States of America
[3] Department of Physics, Arizona State University, Tempe, AZ 85287, United States of America

E-mail: huangl@lzu.edu.cn

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Abstract
Quantum Chaos has been investigated for about a half century. It is an old yet vigorous interdisciplinary field with new concepts and interesting topics emerging constantly. Recent years have witnessed a growing interest in quantum chaos in relativistic quantum systems, leading to the still developing field of *relativistic quantum chaos*. The purpose of this paper is not to provide a thorough review of this area, but rather to outline the basics and introduce the key concepts and methods in a concise way. A few representative topics are discussed, which may help the readers to quickly grasp the essentials of relativistic quantum chaos. A brief overview of the general topics in quantum chaos has also been provided with rich references.

Keywords: quantum chaos, relativistic quantum systems, Dirac billiards

(Some figures may appear in colour only in the online journal)

1. Introduction

Quantum chaos is a branch of fundamental physics investigating the intercapillary field of quantum mechanics, statistical physics, and nonlinear dynamics [1–8]. Even before the establishment of quantum mechanics, in 1913, Bohr proposed quantization rule and used it to successfully predict the energy spectrum of hydrogen atom, which explained the Balmer formula obtained from experimental observations well. Later in 1917, Einstein extended Bohr's quantization rule to integrable systems with global torus structure in phase space [9]. Then he noticed that these quantization rules are only applicable to integrable systems, and would fail for more general, non-integrable systems [9, 10]. About a half century later, in 1970s, inspired by extensive investigations of nonlinear dynamics and chaos, the issue of how to extend the semiclassical quantization rule to non-integrable systems was perceived again by the community, leading to the development of Gutzwiller's trace formula that although being measure zero, the unstable periodic orbits play a crucial rule in shaping the quantum spectral fluctuation behaviors [5, 11–23]. There are quantum systems, e.g. quantum billiards, whose classical counterpart can be chaotic. It is thus mystical that since the Schrödinger equation is linear and thus there is no real chaos in the quantum system, how does it emerge in the semiclassical limit? Note that here 'quantum system' is specifically for the single particle system described by the Schrödinger equation, where many-body effects are excluded. Alternatively, how does the nonlinear and chaotic dynamics that are ubiquitous in the classical world affect the behavior of the corresponding quantum systems? Are there any indicators in the quantum system that can be used to tell whether its semiclassical limit is integrable or chaotic? The efforts to understand these questions and the results constitute the field of quantum chaos, which has attracted extensive attention during the past half century, and has led to profound understandings to the principles of the classical-quantum correspondence [3].

An intriguing phenomenon regarding classical orbits in quantum systems is quantum scar [15, 24–33], where chaotic systems leave behind scars of paths that seem to be retraced in the quantum world [34–40]. Classically, a chaotic system has periodic orbits, but the chaotic nature renders all the orbits being unstable, i.e. an arbitrarily small perturbation to the particle moving along such an orbit could push its motion out of the orbit completely. A paradigmatic model is the two dimensional billiard system, where a particle moves freely inside the billiard, and reflects specularly at the boundary. Thus the shape of the boundary determines the dynamics of the billiard system [8, 41, 42]. A quantum billiard can be constructed similarly, i.e. a two dimensional infinite potential well whose boundary has the same shape of the corresponding classical billiard. Thus in the short wavelength limit, wave dynamics become ray dynamics, and the quantum billiard degenerates to the classical billiard. Classically, the probability to find a particle moving exactly on an unstable periodic orbit is zero, as the measure of these orbits in the phase space vanishes. This leads to the ergodicity of the dynamics. Semiclassically, the averaged Wigner function can be assumed to take the 'microcanonical' form [43], resulting in a Gaussian random function in the coordinate space. Surprisingly, for the quantum counterpart, some of the eigenwavefunctions would concentrate especially around these orbits, forming the quantum scars. The density of a scar along the orbit is a constant and does not depend on $\hbar$ sensitively. But the width of the scars is typically of the order of the wavelength, as $\hbar$ or the wavelength goes to zero, the scar will finally disappear in the random background. 'In this way, the scar 'heals' as $\hbar \rightarrow 0$' [24]. Scars have been searched and analyzed in mesoscopic systems [44–54]. Due to the similarity of the equations for different types of waves, scars have been observed in microwave [55–61], optical fiber and microcavity [62–66], acoustic and liquid surface wave systems [67–70]. Quantum scars in phase space could reveal more information about the classical orbit [24] and have been discussed in [35, 71–77], with numerical evidence of antiscars provided in [76]. The statistical properties of scars has been discussed in [78]. Many-body effects in billiard models have been investigated using the Kohn and Sham (KS) equations in the mean-field approximation, i.e. noninteracting particles moving in some fictitious effective field, and scarring could take place when the disorder is weak and the electron density is sufficiently high [30]. For a comprehensive analysis of quantum scars, please see [79, 80]. Besides the conventional situations, scars on quantum networks are found to be insensitive to the Lyapunov exponents [81]. Quantum scars have also been identified by accumulation of atomic density for certain energies in spin–orbit-coupled atomic gases [82], and observed in the two-dimensional harmonic oscillators due to local impurities [83–85]. Quantum many-body scars become a hot topic recently due to the weak ergodicity breaking caused by these states [86–95], which provides a new route to the departure from the eigenstate thermalization hypothesis (ETH) scenario other than many-body localization (MBL). Their analogy in a driven fracton system, namely, dynamical scar, has also been observed [96]. Note that in these investigations, scar states do not relate to classical periodic orbits as in its original setup, but rather a small number of localized states in an otherwise thermalizing spectrum, while in contrast to both ETH where most of the states behave like thermal states, and MBL in which essentially all eigenstates are athermal.

Another cornerstone of quantum chaos is the random matrix theory (RMT), which was mostly developed in 1950s by Wigner [97–99] when dealing with the energy spectrum of complex quantum systems such as the complex nuclei and later in 1960s by Dyson [100–104]. These works form the basis for random matrix theory (see [105–111] for an overview and recent developments). The idea is such that since the interactions are so complex, it could be efficient to approximate the Hamiltonian by a random matrix with elements following certain statistical properties imposed by the symmetry of the system [105–112]. Density distributions of the energy levels are given, and the distributions of the spacings between nearest neighboring levels are investigated extensively due to the findings that they follow different universal functional forms, namely, Poisson [113] or Wigner–Dyson statistics [114], if the corresponding classical dynamics are integrable or chaotic, respectively. In particular, for systems corresponding to classically chaotic dynamics with no additional geometric symmetry, if time reversal symmetry is preserved, the level spacing statistics would follow RMT with Gassian orthogonal ensembles (GOE). If time reversal symmetry is broken, then Gassian unitary ensembles (GUE) would apply. Gassian symplectic ensembles (GSE) would also appear if the system possesses symplectic symmetry. Long range correlations, i.e. the number variance Σ_2 and spectral rigidity Δ_3, and higher order correlations in spectra are also found to have distinct behavior for quantum systems with integrable or chaotic classical dynamics [106, 115, 116]. Various numerical and experimental evidences are provided [117–121]. Through a series of works [115, 122–125], a connection between classical periodic orbits in chaotic systems and the spectral correlation of the corresponding quantum system represented by the form factor was established, laid the foundation of the universality of the spectral statistics for classically chaotic systems in RMT, which has also been extended into spin $1/2$ [126] and many-body situations [127]. These findings are quite prominent as they could serve as the quantum indications of their classical dynamics and symmetry properties. Note that these statements are for generic systems. Non-generic systems, however, may violate such rules [128]. Level spacing statistics of quantum quasidengeneracy has been investigated and Shnirelman peak was identified [129]. Model for experimental level spacing distributions with missing and spurious levels has also been considered [130, 131]. Generalization of the nearest level spacing statistics to open chaotic wave systems with non-Hermitian Hamiltonian was demonstrated in [132]. Between integrable and chaotic systems, there are pseudo-integrable systems [133], and also mixed dynamical systems with both Kolmogorov–Arnold–Moser (KAM) islands and chaotic sea in the phase space [134]. Non-universal behaviors have been noticed [133–138]. In particular, for singular quantum billiards with a point-like scatterer inside an integrable billiard [137], the level spacing statistics may exhibit intermediate statistics, namely, semi-Poisson [139], that exhibits both level repulsion (in the chaotic case) so the probability to find closeby levels are small, and exponential decay for large spacings as in the Poisson distribution (the integrable case) [140–146]. This feature also appears in quantum systems with parameters close to the metal-insulator transition [147]. While for another class of pseudo-integrable systems, namely, polygonal (particularly triangular) quantum billiards that introduce dividing scattering only at the corners [133, 135], although there are conjectures and numerical discussions [139], the spectral statistic is rather complex and is still an open issue for general cases.

There are many other important topics involving quantum billiards, such as nodal line structures and wavefunction statistics [148–157], quantum chaotic scattering [158–166] with experimental demonstrations on two dimensional electron gas (2DEG) [50, 167–170] where the electrons are described by the two dimensional Schrödinger equation, quantum pointer (preferred) states and decoherence [52, 171–175], universal conductance fluctuations [176–180], chaos-assisted quantum tunneling [181–195], effects of electron–electron interaction [30, 195–199], Loschmidt echo and fidelity [200–236], etc. Being described by the same Helmholtz equation for the spatial wavefunction, e.g. $(\nabla^2 + k^2)\psi = 0$ or its extensions, where ∇^2 is the Laplacian operator and k is the wave number, the quantum billiard can be simulated with other wave systems, such as microwave [55, 59–61, 182, 237–254], light in optical fibers [62, 66] and optical microcavities [255–264], acoustic waves and plate vibrations [68, 265–270], liquid surface waves [67, 69, 70, 271–273], etc.

Other prototypical models that have been investigated extensively in the development of the field of quantum chaos include kicked rotors in terms of diffusion [274–288], entanglement as signatures of referring classical chaos [289–291], experimental realizations [292–295], and other related topics [285, 296–299], and the Dicke model [300–306] and the Lipkin–Meshkov–Glick model [307–310] to account for the many-body effects. In particular, in the phenomena of dynamical localization of kicked rotors with parameters in the classically chaotic region, the momentum localization length has an integer scaling property versus the reduced Planck constant $\hbar$; while in the vicinity of the golden cantori, a fractional $\hbar$ scaling is observed, which was argued as the quantum signature of the golden cantori [311–316]. However, in a following work with a random-pair-kicked particle model, it is found that the fractional $\hbar$ scaling can emerge in systems even without the golden cantori structure at all [296], thus it is not a quantum signature of the classical cantori, but has an origin of inherent quantum nature. Abnormal diffusion in one-dimensional tight-binding lattices [317–322] is another interesting subject which is related to the kicked rotor model, as if the diagonal potential is periodic, it can be mapped into a periodically driven time-dependent quantum problem [276]. Ionization of Rydberg atoms [323–344] and dynamics of Bose–Einstein condensation [197, 345–353] have also been investigated extensively in quantum chaos. Due to the interdisciplinary nature, phenomena and effects in quantum chaos have broad applications in nuclei physics [110, 111, 354, 355], cold atom physics [356–361], controlled laser emission [64, 256, 257, 259, 362–364], quantum information [365–370], etc.

Although being an old field, there are still hot topics and astonishing findings emerging recently due to deeper understandings of the theory, the advances of the computation power and the experimental techniques, such as quantum graphs and their microwave network simulations [131, 371–389], universal quantum manifestations for different classical dynamics [125, 284, 380, 390–396], many-body localization [397–404], quantum thermalization [309, 405–417], quantum thermaldynamics [304, 418–426], out-of-time-ordered correlator (OTOC) [427–436], and other related topics [437, 438]. In particular, there are a number of active groups in China publishing recent works in Communications in Theoretical Physics [166, 207, 236, 285, 340, 359, 439, 440], Chinese Physics B [254, 320, 341, 343, 369, 398, 403, 441–443], Chinese Physics Letters [175, 344, 416, 444, 445], and Science China Physics, Mechanics & Astronomy [288, 446–449].

Among the new developments, one interesting field is to expand the broadly investigated quantum chaos into relativistic quantum systems, and see what happens when the relativistic effect cannot be neglected, e.g. what classical chaos can bring to the relativistic quantum systems. This resulted in the still developing field of relativistic quantum chaos. The first study of relativistic quantum chaos was carried out in 1987, by Berry and Mondragon [450], who invented a two dimensional neutrino billiard by imposing infinite mass confinement on the boundary, i.e. the MIT bag model [451], where the neutrino, with spin $1/2$, at that time was believed to be massless, and was described by the massless Dirac equation. The reason for not choosing the conventional electric potential for confinement is that the Klein tunneling for relativistic particles will invalidate such a confinement. We shall denote such system as *massless Dirac billiards*. The intriguing point is that, with the infinite mass potential, the Hamiltonian breaks the time reversal symmetry, leading to GUE spectral statistics. In a following work by Antoine *et al* [452], a 2D fermionic billiard in a curved space coupled with a magnetic field is considered. Results were obtained under a generalized boundary condition, which confirmed the results by Berry and Mondragon when the boundary condition reduces to the same one. The generalization of the neutrino billiard in a three dimensional cavity has been investigated in [453], with the finding that the orbital lengths seem to be the same as in the scalar spinless case. After Berry and Mondragon's seminal investigation of neutrino (or massless Dirac) billiards [450], there are only a few works in this topic [452, 453]. Only when graphene and other 2D Dirac materials emerged in 2000s [454–460] rendering experimental observation of the effects possible, the field became prosperous and many different aspects have been investigated extensively, e.g. graphene/Dirac billiards were proposed [461, 462] and has started the enthusiasm in this field with either graphene, or microwave artificial graphene (microwave photonic crystal with honeycomb lattice), or by directly solving the massless Dirac equation in a confined region [463, 464].

Since there are different approaches for *relativistic quantum chaos*, before we proceed further, we shall define the boundary of the our discussions clearly to avoid confusion.

First, it should be distinguished from *relativistic chaos*, where the motion of the particle is in relativistic regime, i.e. its speed is comparable to the speed of light, but is described by the classical dynamics, not quantum mechanics. There are many interesting results in this topic [465–479], but it is not considered as relativistic *quantum* chaos. Secondly, quantum chaos has a great motivation regarding classical-quantum correspondence. While for relativistic quantum systems, there are quantities such as spin that does not have a classical correspondence. However, for a relativistic quantum systems and its 'corresponding' classical counterpart by just considering the 'trajectory' of the particle, the properties of the former can be affected significantly by the classical dynamics, i.e. whether chaotic or integrable, of the latter. Therefore, studies of this field are to reveal how classical dynamics may have influence to the 'corresponding' relativistic quantum systems, not to demonstrate the one-to-one correspondence of the two limiting cases. In this sense, there are studies of the relation between entanglement and classical dynamics [289–291, 353, 443, 445, 480–483], spin transport versus classical dynamics [484–489], that although there are no direct one-to-one correspondence, there are significant influence to the behavior of entanglement and spin transport from classical dynamics. Thirdly, the term relativistic quantum chaos is actually not new, but was proposed explicitly about three decades ago in a paper by Tomaschitz entitled 'Relativistic quantum chaos in Robertson–Walker cosmologies' [490]. In this work, Tomaschitz found localized wave fields, which are solutions of the Klein–Gordon equation, quantized on the bounded trajectories in the classical geodesic motion. Actually there is a series of works on this line in quantum cosmology [490–498] concerning chaotic quantum billiards in the vicinity of a cosmological singularity in quantum cosmology, where the local behavior of a part of the metric functions can be described by a billiard on a space of constant negative curvature, leading to the formation of spatial chaos. These results could be helpful to understand the early stages of the Universe. Another line is to examine the spectral properties of the quantum chromodynamics (QCD) lattice Dirac operator [499–507] and Dirac operator on quantum graphs [508, 509], where agreement with chiral random matrix theory has been confirmed.

While our focus has been on billiard systems, kicked rotor was an important model for this field and is still an active research topic [442, 510–512]. As a side note, looking for semiclassical treatment of quantum spinor particles has been a persistent effort from 1930s by Pauli [513] to early of this century [514–531]. These semiclassical results provide insights in understanding the spectral fluctuations in graphene nano-structures [532, 533].

2. Spectral statistics

The main results in level spacing statistics are as follows. For a system with energy levels $\{E_n, n = 1, 2, \cdots\}$, let $\widetilde{N}(E)$ be the number of levels below E. Generally, the density of the spectra is not uniform, therefore, to make comparison of level spacings meaningful, the spectra needs to be unfolded: $x_n \equiv \langle \widetilde{N}(E_n)\rangle$, where $\langle \widetilde{N}(E)\rangle$ is the smooth part of $\widetilde{N}(E)$. Then in general the statistics of x_n follow universal rules depending only on the symmetry of the original quantum system and the corresponding classical dynamics [113, 114], not on the details of the systems. One important quantity is the level spacing distribution $P(S)$, which is the distribution function of the nearest-neighbor spacing, e.g. $S_n = x_{n+1} - x_n$, of the unfolded spectrum $\{x_n\}$. Another quantity is the spectral rigidity $\Delta_3(L)$, for detailed calculations please refer to page 5 of [534] and references therein.

2.1. Berry and Mondragon's result revisited

Berry and Mondragon investigated two dimensional billiard with the African shape (guaranteeing chaotic dynamics with no geometric symmetry) of confined massless spin-1/2 particles, and found GUE statistics [450]. This result is quite surprising as there is no magnetic field or magnetic flux in the system which are typically required to break the T-symmetry. Here the time reversal symmetry is broken by the confinement boundary. The Hamiltonian is given by

$$\hat{H} = -\mathrm{i}\hbar v \hat{\boldsymbol{\sigma}} \cdot \boldsymbol{\nabla} + V(\boldsymbol{r})\hat{\sigma}_z, \tag{1}$$

where v is the Fermi velocity for quasiparticles or the speed of light for a true massless relativistic particle, $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y)$ and $\hat{\sigma}_z$ are Pauli matrices, and $V(\boldsymbol{r})$ is the infinite-mass confinement potential, i.e. $V(\boldsymbol{r}) = 0$ for $\boldsymbol{r}$ inside the billiard region D, while $V(\boldsymbol{r}) = \infty$ otherwise. The time-reversal operator is given by $\hat{T} = \mathrm{i}\sigma_y\hat{K}$, where $\hat{K}$ denotes complex conjugate. It can be readily verified that $\hat{T}\hat{H}\hat{T}^{-1} = -\mathrm{i}\hbar v\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{\nabla} - V(\boldsymbol{r})\hat{\sigma}_z \neq \hat{H}$, i.e. the free motion of the particle is unchanged, but the confinement potential changes sign and breaks T-symmetry. Microscopically, since the spin is locked with the momentum, at each reflection there will be an extra phase due to the rotation of the spin. While for periodic orbits, if the period, or the number of bouncings at the boundary N, is even, then the accumulated phase along the orbit counterclockwisely and clockwisely are the same modulo 2π, thus both orientations will satisfy the quantization rule simultaneously, i.e. if one orientation is a solution of the system, the other orientation (time-reversed) will also be a solution. This is the same for non-relativistic quantum billiards without magnetic field. However, if N is odd, then the accumulated phase difference for two opposite orientations will be π modulo 2π. Thus if one orientation satisfies the quantization condition, i.e. the overall accumulated phase along the complete orbit is integer multiples of 2π, and is thus a solution of the system, the reversed orientation will have an extra π phase and will not satisfy the quantization condition, thus will not be a solution. This breaks the T-symmetry as it requires that the two orientations must be or be not solutions of the system simultaneously.

Although this effect is quite subtle, it can result in GUE, instead of GOE, spectral statistics [450], which has also been verified by solving the system using other numerical techniques such as direct discretization [536], conformal mapping [535], and extended boundary integral method [537] (see also figure 1).

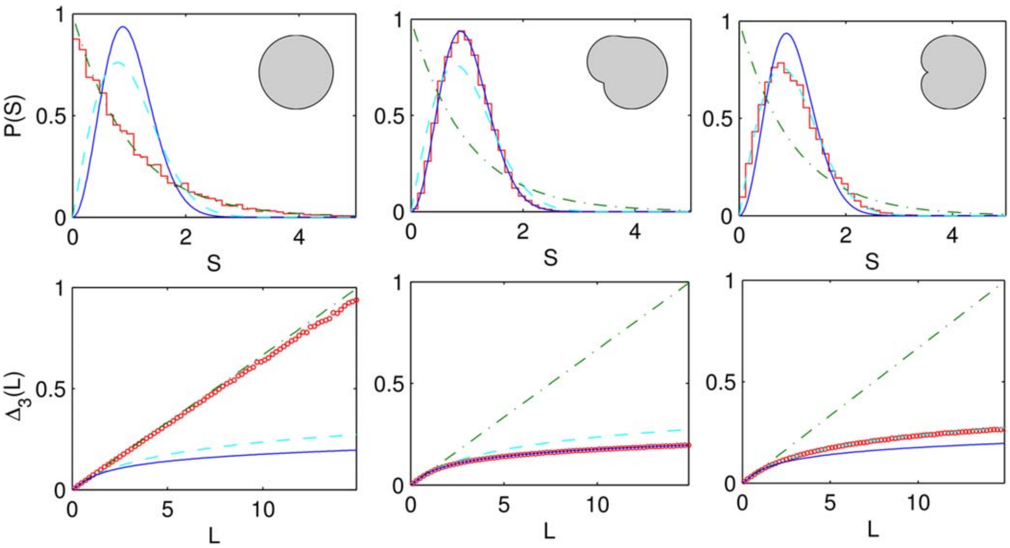

Figure 1. Level spacing statistics of the massless Dirac billiards with the boundary being a circle, the Africa shape, the heart shape for left, middle, and right, respectively. The first row shows the unfolded level-spacing distribution $P(S)$, and the second row shows the spectral rigidity $\Delta_3(L)$. The green dashed–dotted line, cyan dashed line, and blue solid line are for Poisson, GOE, and GUE. Red staircase curves and symbols are numerical results from 13000 energy levels for each shape by diagonalizing the operator $\hat{H}$ given by equation (1). Adapted from [535] with permission.

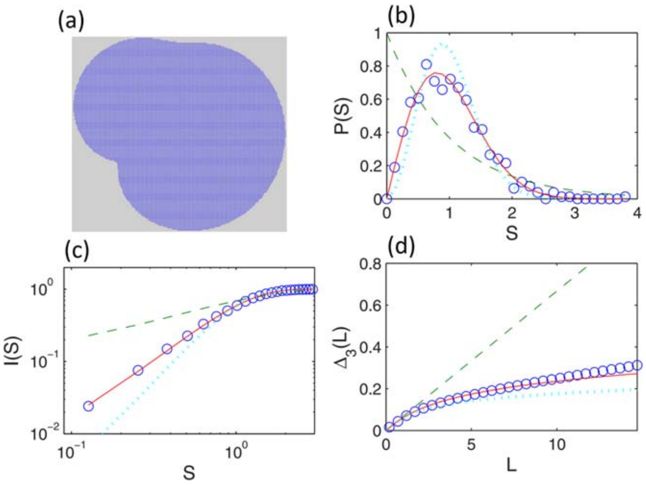

Figure 2. (a) Chaotic graphene billiard with Africa shape cut from a graphene sheet. The system has 42 505 carbon atoms. The outline is determined by the equation $x + iy = 70a\,(z + 0.2z^2 + 0.2z^3 e^{i\pi/3})$, where z is the unit circle in the complex plane, $a = 2.46$ Å is the lattice constant for graphene. The area is $A = 1117\ \mathrm{nm}^2$. (b)–(d) are the level spacing distribution, integrated level spacing distribution, and the spectral rigidity, respectively, for 664 energy levels in the range $0.02 < E_n/t < 0.4$, where t is the hopping energy between nearest neighboring atoms. Dashed line is Poisson, solid line is GOE, and dotted line is GUE. The results show clear evidence of GOE. Adopted from [539] with permission.

Figure 1 shows the results for three billiards: the circular, the African, and the heart-shaped billiards. The circular billiard is integrable, leads to Poisson statistics. The African billiard is chaotic, leads to GUE. The heart-shaped billiard has a mirror symmetry, although it is not symmetric under the time-reversal operation for the corresponding Dirac billiard, it is symmetric under the joint parity and time-reversal operations. Thus again the GOE statistics are recovered. Since the pseudoparticles in graphene follow the same 2D massless Dirac equation as in [450], it is quite natural to ask whether the graphene billiard follow the same GUE statistics. In this regard, the experimental work [462] by counting the resonance peaks in the transport measurement as approximations of intrinsic energy levels, obtained GUE statistics. However, subsequent numerical calculations provide concrete results of GOE statistics in chaotic graphene billiards in the absence of magnetic fields [534, 538, 539], see figure 2, which lead to further experimental investigations using artificial graphene with much higher accuracy and confirmed GOE statistics [540].

2.2. Chaotic graphene billiard

Numerically, the graphene billiard is a graphene sheet where the boundary is cut following a specific shape that carries desired classical dynamics. This is effectively an infinite potential well on the graphene sheet: the potential on the boundary is infinite, and the probability to find an electron on the boundary is zero. The general tight-binding Hamiltonian is given by

$$\hat{H} = \sum(-\varepsilon_i)|i\rangle\langle i| + \sum(-t_{ij})|i\rangle\langle j|, \tag{2}$$

where i and j are the indices of the atoms (or lattice sites), the first summation is over all the atoms within the billiard and the second summation is over pairs of all necessary neighboring atoms, which could be the nearest neighboring pairs, or the next or next–next nearest neighboring pairs, with their respective hopping energy t_{ij}'s. Note that hopping energies between atoms close to the boundary may be different from those far from the boundaries. For clean graphene the onsite energy ε_i is identical for all atoms, thus it is convenient to set it to zero. If there are static electric disorders, ε_i will be position dependent. In the atomic (or lattice site) basis $|j\rangle$, the Hamiltonian matrix element can be calculated as $H_{ij} = \langle i|\hat{H}|j\rangle$, which is given by $(-\varepsilon_i)$ for the diagonal element H_{ii} and $(-t_{ij})$ for element H_{ij}. Once the Hamiltonian matrix is obtained, it can be diagonalized to yield the eigenenergies and the eigenstates. The results of spectral statistics for the African shaped graphene billiard is shown in figure 2, which assumes $\varepsilon_i = 0$ and uniform hopping energies $t_{ij} = t$ between only the nearest neighboring atoms. Thus the energy is in units of t, and it is convenient to use E_n/t for the values of the eigenenergies. It is clear that they follow GOE statistics. Non-idealities such as interactions beyond the nearest neighbors, lattice orientation, effect of boundary bonds and staggered potentials caused by substrates, etc. may have influence to the details of the system, but the GOE statistics are robust and persistent in these non-ideal situations [534].

This might be counterintuitive as one would expect that the graphene chaotic billiards should exhibit the same GUE level-spacing distribution as the massless Dirac billiard [450], since they obey the same equation. The reasoning is as follows. Graphene has two non-equivalent Dirac points (valleys). Quasiparticles in the vicinity of a Dirac point obey the same massless Dirac equation, but the abrupt edge termination in graphene billiard couples the two valleys. As a result, a full set of equations taking into account the effects of both the two nonequivalent Dirac points and the boundary conditions are thus necessary to describe the motion of the relativistic particle. The time-reversal operation for the massless Dirac particle interchanges the two valleys. Thus as a whole, the time-reversal symmetry is preserved [456], resulting in GOE statistics.

Spectral statistics of disordered graphene sheets has also been investigated extensively with both experiments [541] and numerical simulations [542–544], where GOE statistics have been identified in general. Reference [545] examined the level spacing statistics for the edge states only for energies close to the Dirac point. Since these states are localized, it was expected that the statistics may follow that of Poisson, but it turned out that the level spacing statistics was GOE, which can be attributed to the chiral symmetry that introduced long-range correlation between the edge states on different sides, and thus level repulsion. Indeed, when the symmetry is broken by non-zero next nearest neighbor hopping energies, the level spacing statistics becomes Poisson.

Much effort has been devoted to searching for GUE in graphene billiards, e.g. by decoupling the two valleys. A smooth varying mass term was added in [538], however, GUE statistics were not found, which was attributed to the residual inter-valley scatterings. Indications of GUE statistics were found in triangular graphene billiards with zigzag edges and smooth impurity potentials [544], and with an asymmetric strain [546] due to the induced pseudomagnetic field [547, 548].

In the spectral statistics, there is a series of works employing microwave artificial graphene, e.g. a manmade honeycomb lattice not for electrons, but for microwaves [253, 540, 549–563]. Due to the inherent similarity of the wave equations, the quasiparticles follow the same massless Dirac equation and behave similarly as those in graphene. Especially, the Darmstadt group of A. Richter used superconducting microwave cavities filling in photonic crystals, obtained spectra with unprecedentedly high accuracy, yielding convincing statistics [240, 253, 540, 562, 563].

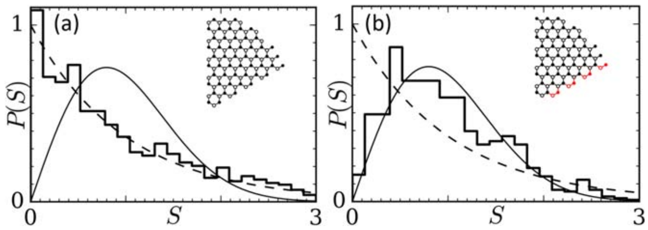

Figure 3. Level spacing statistics of graphene billiard with the shape of a 60° sector with armchair edges. (a) With perfect edges and 227 254 atoms. (b) With one row of atoms removed along one edge so the structure is no longer symmetric (as indicated by the red dots in the inset) and 226 315 atoms. The energy range is $0.02 < E_n/t < 0.2$. Dashed line is Poisson, solid line is GOE. Insets show the magnified view of the lattice structure close to the tip of the sector to illustrate the differences. Adopted from [564] with permission.

2.3. Beyond Berry and Tabor's conjecture

Berry and Tabor [113] proposed that for generically integrable systems, the energy levels are uncorrelated and the resulting statistics would be Poisson. This has been verified by extensive numerical and experimental studies. However, it is found that for graphene billiard with a sector shape where the corresponding classical dynamics are generically integrable, for energy levels close to the Dirac point, the spectral statistics are in general GOE, not Poisson [564]. Only close to the band edge ($E/t = \pm 3$) where the pseudoparticles follow the Schrödinger equation, the statistics become Poisson. The reason for this abnormal phenomenon is that when the energy is close to the Dirac point, the edges play an important role. Even for an ideal situation, say, 60° sector with both straight edges being armchair, the level spacing statistics could be Poisson, as figure 3(a) shows, but changing a few atoms around the tip, or adding or removing one line of atoms along one edge, as demonstrated in the insets, the level spacing statistic becomes that of the GOE (figure 3(b)). Thus the system is extremely sensitive to the imperfections of the boundary, and for sectors with arbitrary angles, the results are generally GOE [564]. An interesting question is that, is this result due to the particular lattice structure of graphene, or due to relativistic nature? A preliminary examination reveals that this might be caused by the complex boundary condition provoked by the multi-component spinor wavefunction. Concrete conclusion may require further investigation.

3. Quantum scars

Quantum scar has been an important pillar for quantum chaos. In the development of relativistic quantum chaos, one natural question is whether scars exist in relativistic quantum systems, and if so, are there any unique features that can distinguish them from the conventional quantum scars?

3.1. Relativistic quantum scars

The existence of scars in relativistic quantum systems was confirmed with a stadium shaped graphene billiard [565]. Scars in the Wimmer system (distorted circular) filled with

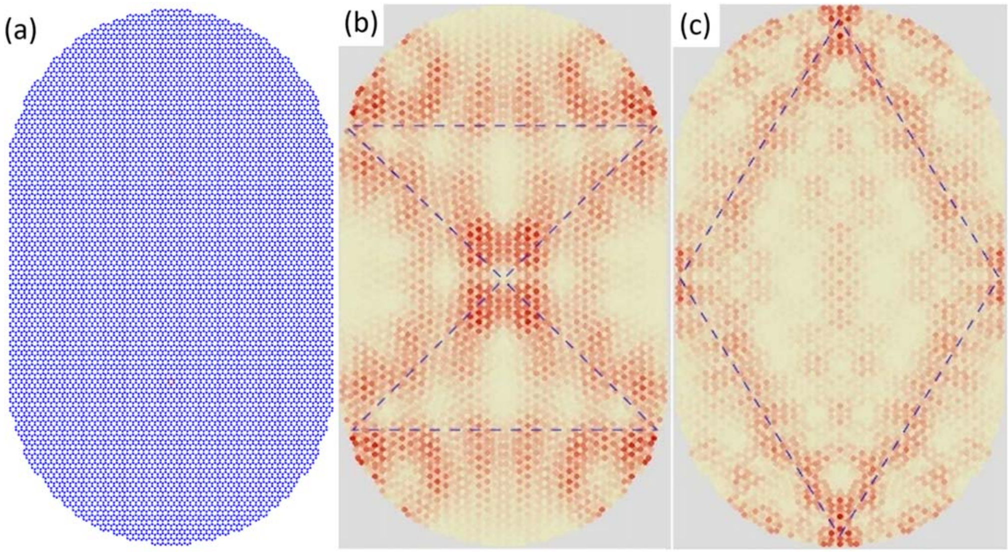

Figure 4. (a) A stadium shaped graphene billiard with 11814 atoms. (b) and (c) show $|\psi_n|^2$ with $E_n/t = 0.363\,58$ and 0.576 65, respectively. Adopted from [565] with permission.

graphene was also observed [566]. Employing the tight-binding Hamiltonian equation (2), the eigenstates ψ_n can be calculated. By examining the spatial distribution of $|\psi_n|^2$ for eigenstates close to Dirac point, unequivocal scars on periodic orbits are observed (figure 4). The scarring state can be formed when the particle, after traveling the orbit for a complete cycle, gains a global phase that is an integer multiple of 2π. Thus when there is a scar occurring at the wavenumber k_0, as the wavenumber (or energy) is changed, there will be a scar again at (or close to) wavenumber k if $\Delta k \cdot L = 2n\pi$, where $\Delta k = k - k_0$, L is the length of the orbit and n is an integer. For two adjacent scarring states, one has $\Delta k = 2\pi/L$. This holds for both massless relativistic and non-relativistic quantum systems. Note that this does not hold for *massive* relativistic quantum billiard systems, as when varying k (or energy E), besides the $\Delta k \cdot L$ term, there will be an additional term that would lead to an extra phase depending on k, which would also need to be taken into account in the quantization formula. For massless relativistic and non-relativistic quantum systems, the key difference lies in the dispersion relation, with $E \propto k$ for the former and $E \propto k^2$ for the latter. Therefore, in terms of E, it will be either E or $\sqrt{E}$ that will be equally spaced for recurrent scars, corresponding to massless relativistic (graphene) or nonrelativistic quantum cases. Figure 5 shows that for two representative scars as shown in the insets, the energy values where they occur versus the relative index. Despite small fluctuations, the linear relation is apparent, corroborating the massless relativistic predictions. In particular, for graphene, since $E = \hbar v_F k$, where $v_F = \sqrt{3}\,ta/(2\hbar)$ is the Fermi velocity, t is the hopping energy between the nearest neighbors, $a = 2.46$ Å is the lattice constant, one has $\Delta E = \hbar v_F \Delta k = h v_F/L$. For the scar shown in the left inset, the length of the orbit is $263a$, yielding $\Delta E = 0.0207t$; while from figure 5, the average ΔE equals to $0.0203t$, which agrees well. For the other scar, the length of the orbit is $275a$,

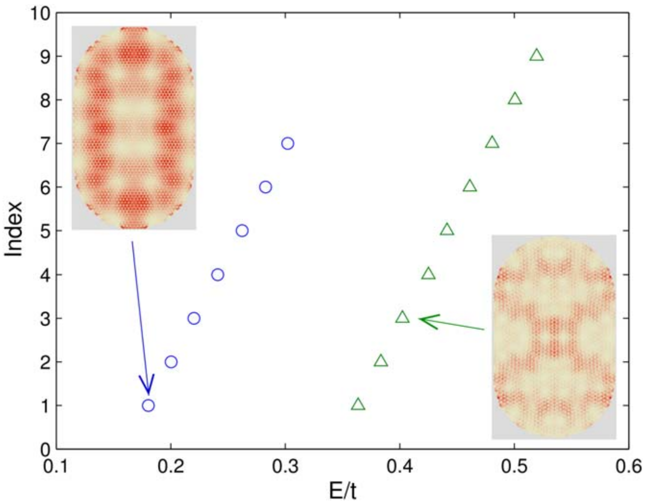

Figure 5. For two representative scars with orbit length $263a$ (left) and $275a$ (right), the energy values where it appears. Vertical axis shows the relative index of these scars. From [464] with permission.

leading to $\Delta E = 0.0198t$, agrees well with $0.0195t$ from figure 5.

Note that only when the energy is small, the dispersion relation is homogeneous and the pseudoparticles follow the massless Dirac equation. When the energy is large, the dispersion relation is no longer homogeneous but direction dependent, and the group velocity $\nabla_k E$ is concentrated in only three directions according to the symmetry of the honeycomb lattice. In this case, the motion of the pseudoparticles deviates from the massless Dirac equation. However, along these three directions, E is still approximately linear to $|\boldsymbol{k}|$ even for E close to t. Since the scars are also constrained on orbits that are composed by straight lines only in these three directions, the relation $\Delta E = h v_F/L$ holds for almost the whole range from 0 to t. This makes it easier to examine the scars and to verify this relation in experiments. Indeed, this

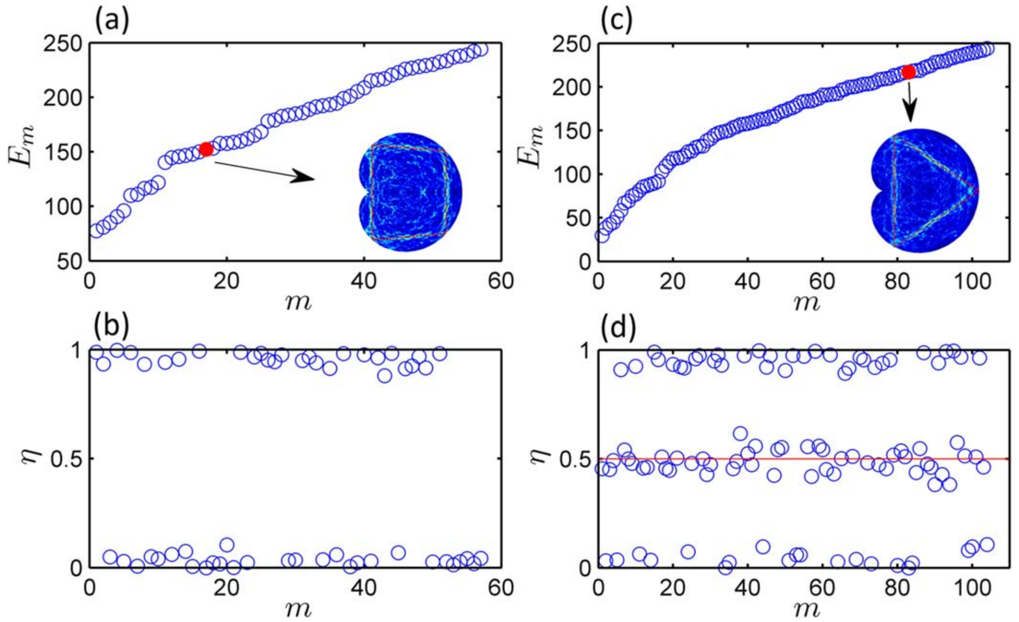

Figure 6. For scars on two representative orbits, period-4 for the left panels, and period-3 for the right panels, the upper panels show the corresponding eigenenergies of the scars, and the lower panels show η (see text) for these sates. Adopted from [568] with permission.

feature of equal spacing of E in the recurring scarring states has been confirmed experimentally in a mesoscopic graphene ring system [567].

3.2. Chiral scars

Although pseudoparticles close to one Dirac point in graphene and Berry and Mondragon's 'neutrino' follow the same 2D massless Dirac equation, due to the coupling of the two Dirac points by the boundary, a complete description for the pseudoparticles in graphene will be different. Thus it is still intriguing to examine the scars in the 'neutrino' billiard and see how the time-reversal symmetry broken by the infinite mass boundary condition is revealed in scars. A direct discretization method was developed to solve the massless Dirac billiard in a confined region, where scars in an African billiard and a bow-tie shaped billiard were identified [536], but due to limited spatial resolution, recurrent rhythm can not be determined. Later, a conformal mapping method was developed where a huge number of eigenstates with extremely high spatial resolution can be obtained [568]. By solving the eigenproblem of a heart-shaped 2D massless Dirac billiard, scars on periodic orbits are identified. Furthermore, it is found that the properties of the scars depend on whether the orbit has even or odd bounces at the boundary, and the relation $k - k_0 = 2\pi n/L$ for recurring scars is no longer fulfilled for the odd orbits. Particularly, for a given reference point k_0 with pronounced scarring patterns, let $\delta k = 2\pi/L$ and define $\eta(m) = (k_m - k_0)/\delta k - [(k_m - k_0)/\delta k]$, where $[x]$ denotes the integer part of x and k_m is the eigenwavenumber of the mth identified scarring state on the same orbit, if the above relation is satisfied, then numerically, $\eta(m)$ will take values that are either close to zero or close to one. As shown in figure 6, this is indeed the case for period-4 orbits. But for period-3 orbits, η takes an extra value close to $1/2$ [568].

A complete understanding would involve many more details [569]. Here we would only provide the main arguments. The quantization condition is such that, following the orbit, after a complete cycle, the total phase accumulation should be integer multiples of 2π. For the massless Dirac billiard with a magnetic flux $\alpha\Phi_0/2\pi$ ($\Phi_0 \equiv h/e$ is the magnetic flux quanta) at the center of the billiard, the total accumulated phase after one complete cycle is

$$\Phi^{\pm} = \frac{1}{\hbar}S - \frac{\sigma\pi}{2} + \beta^{\pm},$$

where '$\pm$' indicates whether the flow of the orbit is counterclockwise or clockwise. $\beta^{\pm} = \sum_i \delta_i^{\pm}$ is the extra phase due to spin rotation imposed by reflections at the boundary, e.g. see figure 7. For a given periodic orbit, at each reflection point, the angle $\delta_i^{\pm}$ can be calculated explicitly, which determines $\beta^{\pm}$ unsuspiciously [569]. The Maslov index σ is the number of conjugate points along the orbit and is canonically invariant [17]. For the heart-shaped chaotic billiard, the value of σ is nothing but the number of reflections along a complete orbit [19]. The action is

$$S = \oint \boldsymbol{p} \cdot \mathrm{d}\boldsymbol{q} = \hbar \oint \boldsymbol{k} \cdot \mathrm{d}\boldsymbol{q} + e \oint \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{q} = k \cdot L \pm W\alpha,$$

where W is the winding number of the orbit with respective to the flux, i.e. how many times it circulates the flux. One thus has

$$\Phi^{\pm} = k \cdot L \pm W\alpha - \frac{\sigma\pi}{2} + \beta^{\pm}. \tag{3}$$

For semiclassically allowed states, the phase accumulation around one cycle should be an integer multiple of 2π, i.e. $\Phi^{\pm} = 2\pi n$ ($n = 1, 2, \cdots$) so as to ensure that the wavefunction is single-valued. One thus has

$$k^{\pm} = (2\pi n \mp W\alpha + \frac{\sigma\pi}{2} - \beta^{\pm})/L. \tag{4}$$

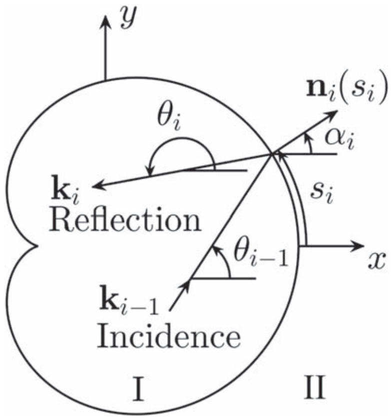

Figure 7. Definition of the angles. $\delta_i^+ = (\theta_i - \theta_{i-1})/2$ is the extra phase due to the rotation of the spin, where '+' indicates counterclockwise orientation. Typically, the phase associated with the time reversed reflection, e.g. δ_i^-, from $-\boldsymbol{k}_i$ to $-\boldsymbol{k}_{i-1}$, would be different from δ_i^+.

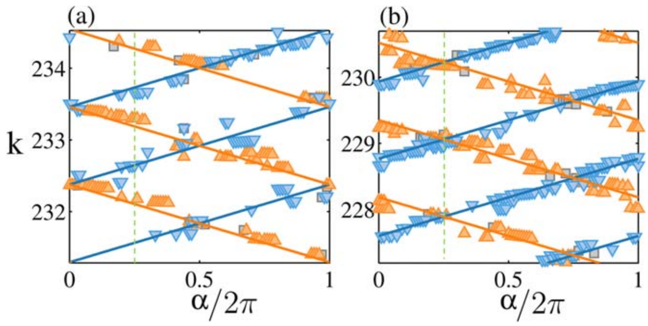

Figure 8. Validation of the quantization rule equation (4). Shown are the relations between wavenumber k and magnetic flux α, for (a) the period-4 scar in figure 6(a), and (b) the period-3 scar in figure 6(c). The orange up-triangles indicate scars with a counterclockwise flow, and the blue down-triangles are those with a clockwise flow. The gray squares mark the scars whose flow orientations cannot be identified, which typically occur close to the cross points of the two orientation cases. The solid lines are theoretical predictions of equation (4). Vertical lines indicate the position of $\alpha = \pi/2$. The step in the variation of α is 0.01. Adapted from [569] with permission.

This is the quantization rule for a scarring state on periodic orbit with length L, which tells at (or close to) which value of wavenumber k (or E) a scar can form. A comparison between this formula and numerical results for two representative orbits are shown in figure 8, which shows good agreement. The time-reversal symmetry is then imprinted with whether there are integers for both counterclockwise and clockwise orientations that could satisfy equation (4) simultaneously for the same value of k, and thus both states with counter-clockwise and clockwise local current flows are solutions of the system. This requires $\Delta\Phi \equiv \Phi^+ - \Phi^- = 0$ modulo 2π, or $2W\alpha + \Delta\beta = 0$ modulo 2π, where $\Delta\beta = \beta^+ - \beta^-$. For systems without a magnetic flux, $\alpha = 0$, the condition becomes $\Delta\beta = 0$. It is surprising that $\Delta\beta$ only depends on whether the periodic orbit has even or odd number of bounces at the boundary: $\Delta\beta = 0$ modulo 2π for even orbits—orbits with even number of bounces, and $\Delta\beta = \pi$ modulo 2π for odd orbits. Thus although each reflection breaks the time-reversal symmetry due to the polarization of the spin and the tangential current at each reflection point [569], even orbits, when considering the overall accumulated phase, preserve the T-symmetry, thus only odd orbits lead to T-symmetry broken. This provides an understanding of the behaviors of η in figure 6. For the period-4 orbit, at $\alpha = 0$, the values of k for scars with different orientation coincide with each other, as shown in figure 8(a) thus they are both allowed when k satisfies the quantization rule. The difference in neighboring k is then $2\pi/L$, leading to η to be either close to 1 or close to 0. However, at $\alpha = 0$, for the period-3 orbit, the values of k for scars with different orientation are interlaced, e.g. clockwise, counterclockwise, clockwise, and so on, as shown in figure 8(b), while for each orientation, the space between neighboring k is $2\pi/L$, but if one does not differentiate the orientations, the difference becomes π/L for the two neighboring k values corresponding to different orientations, leading to $\eta = 1/2$. Note that as the magnetic flux is varied, the system is periodic with $\alpha = 2\pi$. Furthermore, since $\Delta\Phi = 2W\alpha + \Delta\beta$, for period-4 orbit, $\Delta\beta = 0$, $W = 1$, thus when $\alpha = \pi/2$, as indicated by the vertical lines in figure 8, $\Delta\ \Phi$ will become π, which will be similar as that for period-3 at $\alpha = 0$. On the other hand, for period-3 orbit, $\Delta\beta = \pi$, thus when $\alpha = \pi/2$, $\Delta\ \Phi = 2\pi$, or 0 modulo 2π, which is similar to the case of periodic-4 at $\alpha = 0$. Thus by applying a magnetic flux of $\alpha = \pi/2$, the chirality interchanges for these two orbits.

3.3. Unification of chiral scar and nonrelativistic quantum scars

Recently we have developed quantization rule for scars in massive 2D Dirac billiards with infinite mass confinement [570]. Compare to the massless case, there is a new phase emerging during each reflection j. For the massless case, the reflection coefficient R_j is 1. While for the massive case, although the module of R_j is still 1, it has a non-trivial phase, i.e. $R_j^\pm = \mathrm{e}^{\mathrm{i}(\delta_j^\pm + 2\omega_j^\pm)}$, where $\delta_j^\pm = (\theta_j^\pm - \theta_{j-1}^\pm)/2$ is the same as in the massless case (figure 7), but $2\omega_j^\pm$ is a complicated function of the angles (θ_{i-1}, θ_j), the mass m, and the wave-number k (or energy E) [570]. Let $\gamma^\pm = \sum_j 2\omega_j^\pm$ and in the absence of magnetic flux, i.e. $\alpha = 0$, the total phase accumulation around one complete cycle is then

$$\Phi^\pm = k \cdot L - \frac{\sigma\pi}{2} + 2\beta^\pm + \gamma^\pm. \tag{5}$$

Since mod($\Delta\ 2\beta$,2π) = 0, where $\beta^\pm = \sum_j \delta^\pm$, we then have $\Delta\Phi = \Delta\gamma \equiv \gamma^+ - \gamma^-$. Thus for massive Dirac billiards, the complex behavior can be all attributed to $\Delta\gamma$. We have found that when m goes to zero, $\Delta\gamma$ goes to 2π or π for even or odd orbits, respectively (see figure 9 when $m \to 0$), degenerating to the massless cases. When the mass m goes to infinity, $\Delta\gamma$ goes to zero for both even and odd orbits (see figure 9 when $k \to 0$): hence the difference between even and odd orbits diminish and the system becomes effectively a non-relativistic quantum billiard. Thus through the modulation of the extra phase in the reflection coeffecients, the relativistic

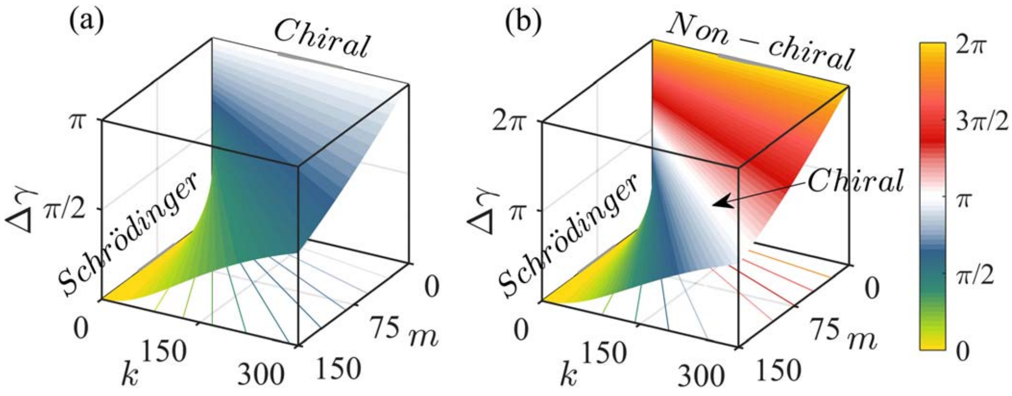

Figure 9. $\Delta\gamma$ between counterclockwise and clockwise scaring states on a period-3 orbit (a) and a period-4 orbit (b). The massless Dirac regime is $m \to 0$, while $k \to 0$ is effectively $m \to \infty$ and is the Schrödinger limit. Adapted from [570] with permission.

chiral scar and the nonrelativistic quantum scar can be unified as the two limiting cases of the massive Dirac billiards.

4. Scattering and tunneling

For open quantum systems, an important topic of quantum chaos is quantum chaotic scattering [50, 158–170]. In nonrelativistic systems, a general observation is that, for classically mixed systems, the transmission (or conductance) of the corresponding quantum system exhibit many sharp resonances caused by the strongly localized states around the classically stable periodic orbits, while for classically chaotic system, the peaks are either broadened or removed. That is, chaos regularizes the quantum transport and makes the transmission curve smoother. Note that, for closed systems, such as a quantum billiard, although there are localized scarring state on the unstable periodic orbits for classically chaotic system, these states are unstable that once the system is opened up, due to the spanning chaotic sea in the phase space, they are typically washed out, leaving few or no localized states.

Similar investigation has been carried out for graphene/Dirac quantum dots with different classical dynamics [538, 571–577]. It has been found that, classical dynamics can indeed influence the quantum transport, e.g. by varying the boundary of the quantum dot to change the corresponding classical dynamics from mixed to chaotic, most of the sharp resonances are broadened or removed, however, there are residual sharp resonances, with still strong localized states on classically *unstable* periodic orbits that would not exist for nonrelativistic systems [573]. In particular, a cosine billiard [163] is adopted to demonstrate this phenomenon. The boundary is given by two hard walls at $y = 0$ and $y = W + (M/2)[1 - \cos(2\pi x/L)]$ for $0 \leqslant x \leqslant L$, with two semi-infinite leads of width W attached at the two openings of the billiard, whose length is L and the widest part is $(W + M)$. By changing the geometric parameters M, W, and L, the classical dynamics can be either mixed, e.g. for $W/L = 0.18$ and $M/L = 0.11$, or chaotic, e.g. $W/L = 0.36$ and $M/L = 0.22$. A tight-binding approach is employed, and Green's function formalism is used to calculate the transmission and the local density of states (LDS) [578–581].

Assume the isolated dot region ($0 \leqslant x \leqslant L$) has Hamiltonian H_c with a set of eigenenergies and eigenfuctions $\{E_{0\alpha}, \psi_{0\alpha}|\alpha = 1, 2,\cdots\}$. The effects of the semi-infinite leads can be incorporated into the retarded self-energy matrices, $\Sigma^R = \Sigma^R_L + \Sigma^R_R$, with the lower indices indicate whether it is due to the left or right leads. Then the whole Hamiltonian with the effects of the leads is $H_c + \Sigma^R$. Since Σ^R in general can be complex, and it is small that it can be regarded as a perturbation, the new set of eigenenergies becomes $E_\alpha = E_{0\alpha} - \Delta_\alpha - \mathrm{i}\gamma_\alpha$, where Δ_α and γ_α are generally small. Δ_α represents a shift in $E_{0\alpha}$, and γ_α is the width of the resonance for the α's state. $1/\gamma_\alpha$ can be regarded as the lifetime of the state [578]. For detailed formulas of calculating γ and the determining factors, please refer to [582].

The values of γ_α for four cases with mixed or chaotic dynamics and 2-dimensional electron gas (2DEG) or graphene quantum dots are shown in figure 10. Note that smaller γ_α will result in sharper transmission resonances. For 2DEG quantum dots with mixed dynamics (figure 10(a)), there are many cases that γ_α takes very small values, in the order of 10^{-4}, indicating extremely sharp resonances. When the classical dynamics change from mixed to chaotic, beside the envelope, the small values in γ_α are almost all removed (figure 10(b)). For graphene quantum dots, when the classical dynamics is mixed, beside the smooth envelope for $\gamma_\alpha \sim 10^{-2}$, there is a cluster of points for small γ_α values (figure 10(c)). When the classical dynamics becomes chaotic (figure 10(d)), although the overall trend is that the small values are shifted upwards, there are still a big cluster of points take apparently smaller values than the envelope, indicating the persistence of the sharp resonances.

In addition, figure 11 shows the LDS for the most pronounced patterns in both the 2DEG and graphene quantum dots. (a) and (d) are for classically mixed dynamics, which show strong localizations on the stable periodic orbits in both cases. (b), (c) and (e), (f) are for classically chaotic dynamics. It is clear that for 2DEG cases, chaos ruined the localized states on the unstable periodic orbits that are present in the closed case, i.e. the scars; but for graphene quantum dot, localization on unstable periodic orbits still persists. Actually,

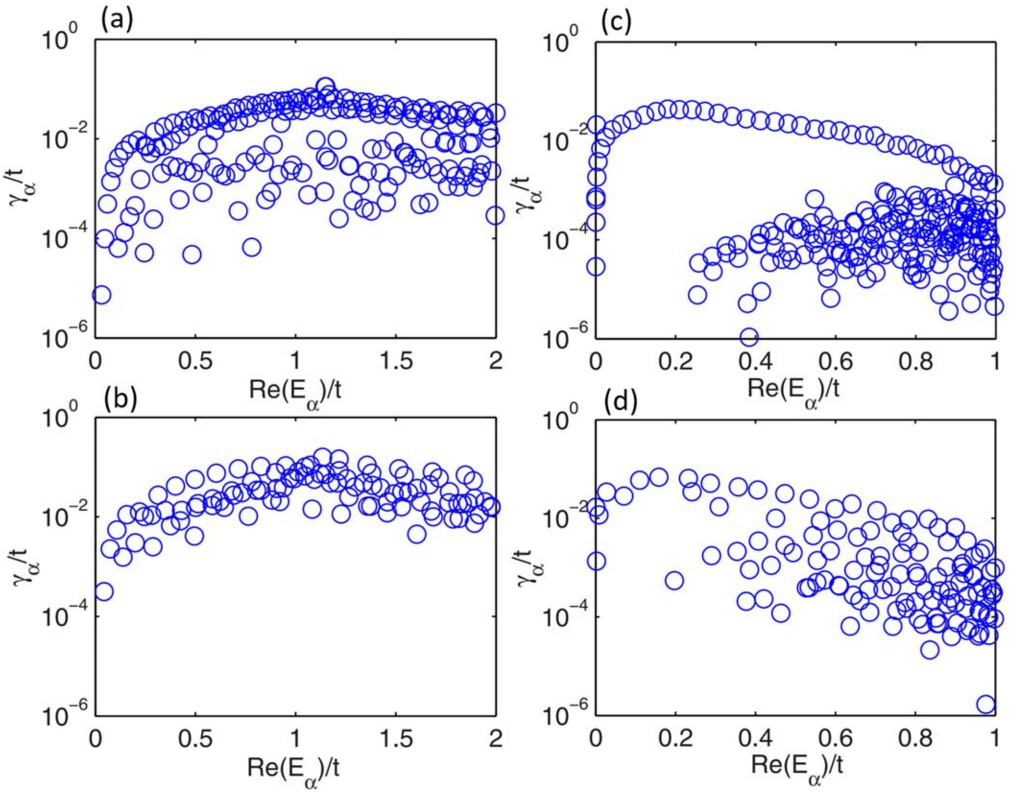

Figure 10. The imaginary part γ of the eigenenergies due to coupling between the dot and the leads, which is an effective indicator of the resonance width. The left panels are for 2DEG quantum dots, and the right panels are for graphene quantum dots. Upper panels are for the cases with mixed dynamics, and lower panels are for classically chaotic dynamics. Adapted from [573] with permission.

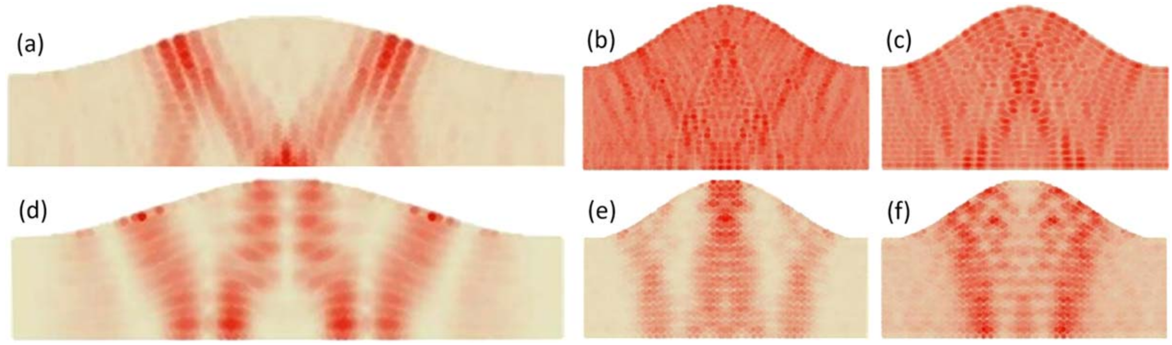

Figure 11. Typical local density of states for quantum dots with classically mixed (a), (d) and chaotic (b) ,(c), (e), (f) dynamics. The upper panels are for 2DEG quantum dots, and lower panels are for graphene quantum dots.

there are many such states, corroborating the results in figure 10. Although weaker, the effect that classical chaos can make the conductance fluctuation becomes smoother can be exploited to articulate a controlling scheme to modulate the conductance fluctuations in quantum transport through a quantum dot, by changing the underlying classical dynamics [583, 584]. In the presence of a strong magnetic field, the difference caused by the classical dynamics can be suppressed further [585].

The same phenomenon has also been observed in bilayer graphene [586]. The pseudoparticles in bilayer graphene follow the 2D massive Dirac equation. Thus this indicates that the suppression of the effect of eliminating sharp resonances by chaos also persists for massive Dirac systems. In addition, when the pseudoparticle is traveling along the classical ballistic orbit, it tends to hop back and forth between the two layers, exhibiting a Zitterbewegung-like effect.

Besides scattering, there are other phenomena that the effect of chaos has been suppressed. For example, for regulation of tunneling rates by chaos [191, 192], it has been found that if 2DEG is replaced by graphene or the massless Dirac fermion, although the regularization effect persist, it is much weaker than the 2DEG case [193–195]. The same hold for persistent currents [587, 588] in Aharonov–Bohm (AB) rings [589]. Conventional metallic [590–593] or semiconductor [594] ring systems with a central AB magnetic flux may exhibit dissipationless currents, e.g. the persistent or permanent current. However, the current is quite sensitive that small non-idealities such as boundary deformation or disorders may destroy the persistent current drastically [595–598]. While in a ring of massless Dirac fermions, due to the Dirac whispering gallery modes [599–601], the persistent current is quite robust against boundary deformations that even in the case where the classical dynamics become chaotic there is still a quite large amount of persistent currents [602–604]. Furthermore, recently, Han *et al* [605] investigated out-of-time-order correlator in relativistic quantum billiard systems and found that the signatures of classical chaos are less pronounced than in the nonrelativistic case. Here again the effect of chaos is suppressed.

5. Quantum chaos in pseudospin-1 Dirac materials

Dirac materials hosting pseudospin-1 quasiparticles with a conical intersection of triple degeneracy in the underlying energy band have attracted a great deal of attention [606–633]. The physics of these 2D Dirac materials is described by the generalized Dirac-Weyl equation for massless spin-1 particles [607, 608, 626]. Pseudospin-1 quasiparticles are different from Dirac, Weyl and Majorana fermions, and are of particular interest to the broad research community with diverse experimental realization schemes such as artificial photonic lattices [612, 616, 620, 621, 624], optical [622] and electronic Lieb lattices [631, 632], as well as superconducting qutrits [633]. A striking relativistic quantum hallmark of pseudospin-1 particles is super-Klein tunneling through a scalar potential barrier [608, 610, 623, 634, 635], where omnidirectional and perfect transmission of probability one occurs when the incident energy is about one half of the potential height. Generally, Klein tunneling defines optical-like, negatively refracted ray paths through the barrier interface via angularly resolved transmittance in the short wavelength limit [636–638].

A recent study [639] addressed the issue of confinement of quasiparticles in pseudospin-1 materials. When both super-Klein tunneling and chaos are present, one may intuitively expect severe leakage to predominantly occur so that trapping would be impossible. However, quite counterintuitively, an energy range was found in which robust wave confinement occurs in spite of chaos and super-Klein tunneling. Especially, the three-component spinor wave concentrates in a particular region of the boundary through strongly squeezed local current vortices generated there, whose pattern in physical space can be manipulated in a reconfigurable manner, e.g. by deforming the boundary shape or setting the direction of excitation wave. While these modes are distributed unevenly in physical space because of the irregular deformations, even fully developed chaos and super-Klein tunneling are not able to reduce their trapping lifetime. That is, these modes contradict the intuitive expectation that electrostatically confining relativistic type of carriers/particles to a finite chaotic domain is impossible due to the simultaneous presence of two leaking (Q-spoiling) mechanisms: chaos assisted tunneling and Klein tunneling. This phenomenon has no counterpart in nonrelativistic quantum or even in pseudospin-$1/2$ systems. The resulting narrow resonances are also characteristically different from those due to scarring modes concentrating on periodic orbits in conventional wave chaotic scattering, in quantum dots [573, 640–645] or in open optical microcavities [646–648].

6. Discussions

Beside the above discussed few topics in relativistic quantum chaos, there are many other interesting topics that have been investigated in depth, such as quantum tunneling without [193, 194] and with electron–electron interactions [195], super-persistent currents that are robust to boundary deformations [602, 603] and the presence of disorders [604], relativistic quantum chimera states that electrons with different spins exhibit distinct scattering behaviors as they follow different classical dynamics [649], OTOC for relativistic quantum systems [605], anomalous entanglement in chaotic Dirac billiards [650], relativistic quantum kicked rotors [442, 510–512], kicked relativistic particle in a box [651], etc. More efforts are needed to gain deeper understandings of these interesting subjects. In addition, electron–electron interaction effects [652] in a chaotic graphene quantum billiard have also been considered and compared with scanning tunneling microscopy (STM) experiments, which could explain both the measured density of state values and the experimentally observed topography patterns [198]. Most of the understandings achieved so far for relativistic quantum chaos are for massless cases. Massive Dirac billiards have been considered only recently, where quantization formula for scarred states in confined 2D massive Dirac billiard has been proposed and validated numerically, and restoration of time-reversal symmetry in the infinite mass limit has been unveiled [570]. However, there are many other issues to be understood in the massive Dirac billiards, e.g. to what extent the intriguing observations for massless Dirac billiard persist in the massive case? Pseudo-spin one systems [607–609] have attracted much attention recently. Due to the flat band, it has many interesting properties regarding quantum chaotic scattering, such as superscattering that could even defy chaos Q-spoiling and Klein tunneling [635, 639, 653, 654]. There are still many open questions concerning pseudo-spin one system and quantum chaos.

Retrospecting the half-century development of quantum chaos, there are many subjects that would be interesting to extend into the relativistic quantum realm, such as the validity of the proposed indicators of universality corresponding to different classical dynamics, Loschmidt echo, many-body effects, quantum thermalization, etc. that have been discussed in section 1, as it is not straightforward to speculate what will happen when stepping into the relativistic regime. Efforts in trying to understand the behaviors of these subjects in relativistic quantum systems may not only advance the knowledge on the fundamental physics of relativistic quantum chaos, but may also bring new concepts of applications base on the state-of-art Dirac material technologies.

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References

[1] Stöckmann H-J 2006 *Quantum Chaos: An Introduction* (New York: Cambridge University Press)
[2] Haake F 2010 *Quantum Signatures of Chaos* (*Springer Series in Synergetics*) 3rd edn (Berlin: Springer)
[3] Gutzwiller M C 2013 *Chaos in Classical and Quantum Mechanics* (New York: Springer)
[4] Xu G 1995 *Quantum Chaotic Motions in Quantum Systems* (Shanghai: Shanghai Scientific and Technical Publishers)
[5] Gu Y 1996 *Quantum Chaos* (Shanghai: Shanghai Scientific and Technological Education Publishing House)
[6] Casati G and Chirikov B 2006 *Quantum Chaos: Between Order and Disorder* (Cambridge: Cambridge University Press)
[7] Knauf A and Sinai Y G 1997 *Classical Nonintegrability, Quantum Chaos* vol 27 (Basel: Birkhäuser)
[8] Cvitanovic P, Artuso R, Mainieri R, Tanner G, Vattay G, Whelan N and Wirzba A 2005 *Chaos: Classical and Quantum, ChaosBook.org* (Copenhagen: Niels Bohr Institute)
[9] Einstein A 1917 Zum quantensatz von Sommerfeld und Epstein *Verh. Dtsch. Phys. Ges.* **19** 82
[10] Stone A D 2005 Einstein's unknown insight and the problem of quantizing chaos *Phys. Today* **58** 37–43
[11] Gutzwiller M C 1971 Periodic orbits and classical quantization conditions *J. Math. Phys.* **12** 343–58
[12] Berry M V and Mount K E 1972 Semiclassical approximations in wave mechanics *Rep. Prog. Phys.* **35** 315–97
[13] Miller W H 1975 Semiclassical quantization of nonseparable systems: a new look at periodic orbit theory *J. Chem. Phys.* **63** 996–9
[14] Berry M and Tabor M 1977 Calculating the bound spectrum by path summation in action-angle variables *J. Phys. A: Math. Gen. Math. Theor.* **10** 371–9
[15] Mcdonald S W and Kaufman A N 1979 Spectrum and eigenfunctions for a Hamiltonian with stochastic trajectories *Phys. Rev. Lett.* **42** 1189–91
[16] Du M L and Delos J B 1987 Effect of closed classical orbits on quantum spectra: ionization of atoms in a magnetic field *Phys. Rev. Lett.* **58** 1731–3
[17] Creagh S C, Robbins J M and Littlejohn R G 1990 Geometrical properties of Maslov indices in the semiclassical trace formula for the density of states *Phys. Rev.* A **42** 1907
[18] Cvitanović P, Rosenqvist P E, Vattay G, Rugh H H and Fredholm A 1993 Determinant for semiclassical quantization *Chaos* **3** 619–36
[19] Bruus H and Whelan N D 1996 Edge diffraction, trace formulae and the cardioid billiard *Nonlinearity* **9** 1023
[20] Primack H and Smilansky U 1998 On the accuracy of the semiclassical trace formula *J. Phys. A: Math. Gen.* **31** 6253–77
[21] Cohen D, Primack H and Smilansky U 1998 Quantal-classical duality and the semiclassical trace formula *Ann. Phys.* **264** 108–70
[22] Yang S and Kellman M E 2002 Perspective on semiclassical quantization: how periodic orbits converge to quantizing tori *Phys. Rev.* A **66** 052113
[23] Müller S, Heusler S, Braun P, Haake F and Altland A 2005 Periodic-orbit theory of universality in quantum chaos *Phys. Rev.* E **72** 046207
[24] Heller E J 1984 Bound-state eigenfunctions of classically chaotic Hamiltonian systems—scars of periodic-orbits *Phys. Rev. Lett.* **53** 1515–8
[25] Mcdonald S W and Kaufman A N 1988 Wave chaos in the stadium—statistical properties of short-wave solutions of the Helmholtz equation *Phys. Rev.* A **37** 3067–86
[26] Wintgen D and Hönig A 1989 Irregular wave functions of a hydrogen atom in a uniform magnetic field *Phys. Rev. Lett.* **63** 1467–70
[27] Kus M, Zakrzewski J and Zyczkowski K 1991 Quantum scars on a sphere *Phys. Rev.* A **43** 4244–8
[28] Malta C P, Deaguiar M A M and Dealmeida A M O 1993 Quantum signature of a period-doubling bifurcation and scars of periodic orbits *Phys. Rev.* A **47** 1625–32
[29] Bellomo P and Uzer T 1995 Quantum scars and classical ghosts *Phys. Rev.* A **51** 1669–72
[30] Agam O 1996 Quantum scars of classical orbits in small interacting electronic systems *Phys. Rev.* B **54** 2607–28
[31] Simonotti F P, Vergini E and Saraceno M 1997 Quantitative study of scars in the boundary section of the stadium billiard *Phys. Rev.* E **56** 3859–67
[32] Li B 1997 Numerical study of scars in a chaotic billiard *Phys. Rev.* E **55** 5376–9
[33] Li B and Hu B 1998 Statistical analysis of scars in stadium billiard *J. Phys. A: Math. Gen.* **31** 483
[34] Bogomolny E B 1988 Smoothed wave-functions of chaotic quantum systems *Physica* D **31** 169–89
[35] Berry M V 1989 Quantum scars of classical closed orbits in phase-space *Proc. R. Soc.* A **423** 219–31
[36] Agam O and Fishman S 1993 Quantum eigenfunctions in terms of periodic orbits of chaotic systems *J. Phys. A: Math. Gen.* **26** 2113–37
[37] Agam O and Fishman S 1994 Semiclassical criterion for scars in wave functions of chaotic systems *Phys. Rev. Lett.* **73** 806–9
[38] Fishman S, Georgeot B and Prange R E 1996 Fredholm method for scars *J. Phys. A: Math. Gen.* **29** 919–37
[39] Eckhardt B, Fishman S, Keating J, Agam O, Main J and Müller K 1995 Approach to ergodicity in quantum wave functions *Phys. Rev.* E **52** 5893–903
[40] Bäcker A, Schubert R and Stifter P 1998 Rate of quantum ergodicity in Euclidean billiards *Phys. Rev.* E **57** 5425–47
[41] Lichtenberg A J and Lieberman M A 1992 *Regular and Chaotic Dynamics* 2nd edn (New York: Springer)
[42] Ott E 2002 *Chaos in Dynamical Systems* 2nd edn (Cambridge: Cambridge University Press)
[43] Berry M V 1977 Regular and irregular semiclassical wavefunctions *J. Phys. A: Math. Gen.* **10** 2083
[44] Fromhold T M, Wilkinson P B, Sheard F W, Eaves L, Miao J and Edwards G 1995 Manifestations of classical chaos in the energy level spectrum of a quantum well *Phys. Rev. Lett.* **75** 1142–5
[45] Wilkinson P B, Fromhold T M, Eaves L, Sheard F W, Miura N and Takamasu T 1996 Observation of 'scarred' wavefunctions in a quantum well with chaotic electron dynamics *Nature* **380** 608–10
[46] Monteiro T S, Delande D and Connerade J P 1997 Have quantum scars been observed? *Nature* **387** 863–4
[47] Akis R, Ferry D K and Bird J P 1997 Wave function scarring effects in open stadium shaped quantum dots *Phys. Rev. Lett.* **79** 123–6
[48] Narimanov E E and Stone A D 1998 Origin of strong scarring of wave functions in quantum wells in a tilted magnetic field *Phys. Rev. Lett.* **80** 49–52

[49] Bird J P, Akis R, Ferry D K, Vasileska D, Cooper J, Aoyagi Y and Sugano T 1999 Lead-orientation-dependent wave function scarring in open quantum dots *Phys. Rev. Lett.* **82** 4691–4
[50] Crook R, Smith C G, Graham A C, Farrer I, Beere H E and Ritchie D A 2003 Imaging fractal conductance fluctuations and scarred wave functions in a quantum billiard *Phys. Rev. Lett.* **91** 246803
[51] LeRoy B J, Bleszynski A C, Aidala K E, Westervelt R M, Kalben A, Heller E J, Shaw S E J, Maranowski K D and Gossard A C 2005 Imaging electron interferometer *Phys. Rev. Lett.* **94** 126801
[52] Brunner R, Akis R, Ferry D K, Kuchar F and Meisels R 2008 Coupling-induced bipartite pointer states in arrays of electron billiards: Quantum Darwinism in action? *Phys. Rev. Lett.* **101** 024102
[53] Burke A M, Akis R, Day T E, Speyer G, Ferry D K and Bennett B R 2010 Periodic scarred states in open quantum dots as evidence of quantum Darwinism *Phys. Rev. Lett.* **104** 176801
[54] Aoki N, Brunner R, Burke A M, Akis R, Meisels R, Ferry D K and Ochiai Y 2012 Direct imaging of electron states in open quantum dots *Phys. Rev. Lett.* **108** 136804
[55] Sridhar S 1991 Experimental observation of scarred eigenfunctions of chaotic microwave cavities *Phys. Rev. Lett.* **67** 785–8
[56] Sridhar S, Hogenboom D O and Willemsen B A 1992 Microwave experiments on chaotic billiards *J. Stat. Phys.* **68** 239–58
[57] Jensen R V 1992 Quantum chaos *Nature* **355** 311–8
[58] Jensen R V 1992 Quantum mechanics—bringing order out of chaos *Nature* **355** 591–2
[59] Stein J and Stöckmann H-J 1992 Experimental determination of billiard wave functions *Phys. Rev. Lett.* **68** 2867–70
[60] Sridhar S and Heller E J 1992 Physical and numerical experiments on the wave mechanics of classically chaotic systems *Phys. Rev.* A **46** R1728–31
[61] Kudrolli A, Kidambi V and Sridhar S 1995 Experimental studies of chaos and localization in quantum wave functions *Phys. Rev. Lett.* **75** 822–5
[62] Doya V, Legrand O, Mortessagne F and Miniatura C 2001 Light scarring in an optical fiber *Phys. Rev. Lett.* **88** 014102
[63] Lee S-B, Lee J-H, Chang J-S, Moon H-J, Kim S W and An K 2002 Observation of scarred modes in asymmetrically deformed microcylinder lasers *Phys. Rev. Lett.* **88** 033903
[64] Gmachl C, Narimanov E E, Capasso F, Baillargeon J N and Cho A Y 2002 Kolmogorov–Arnold–Moser transition and laser action on scar modes in semiconductor diode lasers with deformed resonators *Opt. Lett.* **27** 824–6
[65] Harayama T, Fukushima T, Davis P, Vaccaro P O, Miyasaka T, Nishimura T and Aida T 2003 Lasing on scar modes in fully chaotic microcavities *Phys. Rev.* E **67** 015207
[66] Michel C, Doya V, Legrand O and Mortessagne F 2007 Selective amplification of scars in a chaotic optical fiber *Phys. Rev. Lett.* **99** 224101
[67] Blümel R, Davidson I H, Reinhardt W P, Lin H and Sharnoff M 1992 Quasilinear ridge structures in water surface waves *Phys. Rev.* A **45** 2641–4
[68] Chinnery P A and Humphrey V F 1996 Experimental visualization of acoustic resonances within a stadium-shaped cavity *Phys. Rev.* E **53** 272–6
[69] Kudrolli A, Abraham M C and Gollub J P 2001 Scarred patterns in surface waves *Phys. Rev.* E **63** 026208
[70] Agam O and Altshuler B L 2001 Scars in parametrically excited surface waves *Physica* A **302** 310–7
[71] Waterland R L, Yuan J-M, Martens C C, Gillilan R E and Reinhardt W P 1988 Classical-quantum correspondence in the presence of global chaos *Phys. Rev. Lett.* **61** 2733–6
[72] Jensen R V, Sanders M M, Saraceno M and Sundaram B 1989 Inhibition of quantum transport due to scars of unstable periodic-orbits *Phys. Rev. Lett.* **63** 2771–5
[73] Feingold M, Littlejohn R G, Solina S B, Pehling J and Piro O 1990 Scars in billiards: The phase space approach *Phys. Lett.* A **146** 199–203
[74] Depolavieja G G, Borondo F and Benito R M 1994 Scars in groups of eigenstates in a classically chaotic system *Phys. Rev. Lett.* **73** 1613–6
[75] Arranz F J, Borondo F and Benito R M 1998 Scar formation at the edge of the chaotic region *Phys. Rev. Lett.* **80** 944–7
[76] Wang J, Lai C-H and Gu Y 2001 Ergodicity and scars of the quantum cat map in the semiclassical regime *Phys. Rev.* E **63** 056208
[77] Wisniacki D, Vergini E, Benito R and Borondo F 2006 Scarring by homoclinic and heteroclinic orbits *Phys. Rev. Lett.* **97** 094101
[78] Antonsen T M, Ott E, Chen Q and Oerter R N 1995 Statistics of wave-function scars *Phys. Rev.* E **51** 111–21
[79] Kaplan L and Heller E J 1998 Linear and nonlinear theory of eigenfunction scars *Ann. Phys.* **264** 171–206
[80] Kaplan L 1999 Scars in quantum chaotic wavefunctions *Nonlinearity* **12** R1
[81] Schanz H and Kottos T 2003 Scars on quantum networks ignore the Lyapunov exponent *Phys. Rev. Lett.* **90** 234101
[82] Larson J, Anderson B M and Altland A 2013 Chaos-driven dynamics in spin–orbit-coupled atomic gases *Phys. Rev.* A **87** 013624
[83] Luukko P J J, Drury B, Klales A, Kaplan L, Heller E J and Räsänen E 2016 Strong quantum scarring by local impurities *Sci. Rep.* **6** 37656
[84] Keski-Rahkonen J, Luukko P J J, Kaplan L, Heller E J and Räsänen E 2017 Controllable quantum scars in semiconductor quantum dots *Phys. Rev.* B **96** 094204
[85] Keski-Rahkonen J, Ruhanen A, Heller E J and Räsänen E 2019 Quantum Lissajous scars *Phys. Rev. Lett.* **123** 214101
[86] Turner C J, Michailidis A A, Abanin D A, Serbyn M and Papic Z 2018 Weak ergodicity breaking from quantum many-body scars *Nat. Phys.* **14** 1
[87] Turner C J, Michailidis A A, Abanin D A, Serbyn M and Papić Z 2018 Quantum scarred eigenstates in a Rydberg atom chain: entanglement, breakdown of thermalization, and stability to perturbations *Phys. Rev.* B **98** 155134
[88] Moudgalya S, Regnault N and Bernevig B A 2018 Entanglement of exact excited states of Affleck–Kennedy–Lieb–Tasaki models: exact results, many-body scars, and violation of the strong eigenstate thermalization hypothesis *Phys. Rev.* B **98** 235156
[89] Ho W W, Choi S, Pichler H and Lukin M D 2019 Periodic orbits, entanglement, and quantum many-body scars in constrained models: matrix product state approach *Phys. Rev. Lett.* **122** 040603
[90] Choi S, Turner C J, Pichler H, Ho W W, Michailidis A A, Papić Z, Serbyn M, Lukin M D and Abanin D A 2019 Emergent SU(2) dynamics and perfect quantum many-body scars *Phys. Rev. Lett.* **122** 220603
[91] Lin C-J and Motrunich O I 2019 Exact quantum many-body scar states in the Rydberg-blockaded atom chain *Phys. Rev. Lett.* **122** 173401
[92] Mamaev M, Kimchi I, Perlin M A, Nandkishore R M and Rey A M 2019 Quantum entropic self-localization with ultracold fermions *Phys. Rev. Lett.* **123** 130402
[93] Schecter M and Iadecola T 2019 Weak ergodicity breaking and quantum many-body scars in spin-1 xy magnets *Phys. Rev. Lett.* **123** 147201
[94] Wilming H, Goihl M, Roth I and Eisert J 2019 Entanglement-ergodic quantum systems equilibrate exponentially well *Phys. Rev. Lett.* **123** 200604

[95] Iadecola T, Schecter M and Xu S 2019 Quantum many-body scars from magnon condensation *Phys. Rev.* B **100** 184312
[96] Pai S and Pretko M 2019 Dynamical scar states in driven fracton systems *Phys. Rev. Lett.* **123** 136401
[97] Wigner E P 1955 Characteristic vectors of bordered matrices with infinite dimensions *Ann. Math.* **62** 548–64
[98] Wigner E P 1957 Characteristic vectors of bordered matrices with infinite dimensions ii *Ann. Math.* **65** 203–7
[99] Wigner E P 1958 On the distribution of the roots of certain symmetric matrices *Ann. Math.* **62** 325–7
[100] Dyson F J 1962 Statistical theory of the energy levels of complex systems. I *J. Math. Phys.* **3** 140–56
[101] Dyson F J 1962 Statistical theory of the energy levels of complex systems. II *J. Math. Phys.* **3** 157–65
[102] Dyson F J 1962 Statistical theory of the energy levels of complex systems. III *J. Math. Phys.* **3** 166–75
[103] Dyson F J and Mehta M L 1963 Statistical theory of the energy levels of complex systems. IV *J. Math. Phys.* **4** 701–12
[104] Mehta M L and Dyson F J 1963 Statistical theory of the energy levels of complex systems. V *J. Math. Phys.* **4** 713–9
[105] Wigner E P 1967 Random matrices in physics *SIAM Rev.* **9** 1C23
[106] Mehta M L 2004 *Random Matrices* vol 142 (New York: Academic)
[107] Brody T A, Flores J, French J B, Mello P A, Pandey A and Wong S S M 1981 Random-matrix physics: spectrum and strength fluctuations *Rev. Mod. Phys.* **53** 385–479
[108] Guhr T, Groeling A M and Weidenmüller H A 1998 Random-matrix theories in quantum physics: common concepts *Phys. Rep.* **299** 189–425
[109] Alhassid Y 2000 The statistical theory of quantum dots *Rev. Mod. Phys.* **72** 895–968
[110] Weidenmüller H A and Mitchell G E 2009 Random matrices and chaos in nuclear physics: nuclear structure *Rev. Mod. Phys.* **81** 539–89
[111] Mitchell G E, Richter A and Weidenmüller H A 2010 Random matrices and chaos in nuclear physics: nuclear reactions *Rev. Mod. Phys.* **82** 2845–901
[112] Caselle M and Magnea U 2004 Random matrix theory and symmetric spaces *Phys. Rep.* **394** 41–156
[113] Berry M V and Tabor M 1977 Level clustering in the regular spectrum *Proc. R. Soc.* A **356** 375–94
[114] Bohigas O, Giannoni M J and Schmit C 1984 Characterization of chaotic quantum spectra and universality of level fluctuation laws *Phys. Rev. Lett.* **52** 1–4
[115] Berry M V 1985 Semiclassical theory of spectral rigidity *Proc. R. Soc.* A **400** 229–51
[116] Bohigas O, Haq R U and Pandey A 1985 Higher-order correlations in spectra of complex systems *Phys. Rev. Lett.* **54** 1645–8
[117] Terasaka T and Matsushita T 1985 Statistical properties of the quantized energy spectrum of a Hamiltonian system with classically regular and chaotic trajectories: a numerical study of level-spacing distributions for two-dimensional coupled Morse-oscillator systems *Phys. Rev.* A **32** 538–51
[118] Jiang Y 1988 Level statistics of doublet spectrum of Sinai's billiard *Chin. Phys. Lett.* **5** 541–4
[119] Delande D and Gay J C 1986 Quantum chaos and statistical properties of energy levels: numerical study of the hydrogen atom in a magnetic field *Phys. Rev. Lett.* **57** 2006–9
[120] Fromhold T M, Wilkinson P B, Sheard F W, Eaves L, Miao J and Edwards G 1995 Manifestations of classical chaos in the energy-level spectrum of a quantum-well *Phys. Rev. Lett.* **75** 1142–5
[121] Zhou W, Chen Z, Zhang B, Yu C H, Lu W and Shen S C 2010 Magnetic field control of the quantum chaotic dynamics of hydrogen analogs in an anisotropic crystal field *Phys. Rev. Lett.* **105** 024101
[122] Sieber M 2000 Spectral statistics in chaotic systems with a point interaction *J. Phys. A: Math. Gen.* **33** 6263–78
[123] Sieber M and Richter K 2001 Correlations between periodic orbits and their role in spectral statistics *Phys. Scr.* **T90** 128
[124] Heusler S, Müller S, Braun P and Haake F 2004 universal spectral form factor for chaotic dynamics *J. Phys. A: Math. Gen.* **37** L31–7
[125] Müller S, Heusler S, Braun P, Haake F and Altland A 2004 Semiclassical foundation of universality in quantum chaos *Phys. Rev. Lett.* **93** 014103
[126] Nagao T and Saito K 2007 Semiclassical approach to parametric spectral correlation with spin $1/2$ *J. Phys. A: Math. Gen.* **40** 12055–70
[127] Kos P, Ljubotina M and Prosen T C V 2018 Many-body quantum chaos: analytic connection to random matrix theory *Phys. Rev.* X **8** 021062
[128] Bolte J, Steil G and Steiner F 1992 Arithmetical chaos and violation of universality in energy level statistics *Phys. Rev. Lett.* **69** 2188–91
[129] Chirikov B V and Shepelyansky D L 1995 Shnirelman peak in level spacing statistics *Phys. Rev. Lett.* **74** 518–21
[130] Lehmann K K and Coy S L 1987 The Gaussian orthogonal ensemble with missing and spurious levels: A model for experimental level spacing distributions *J. Chem. Phys.* **87** 5415–8
[131] Białous M, Yunko V, Bauch S, Ławniczak M, Dietz B and Sirko L 2016 Power spectrum analysis and missing level statistics of microwave graphs with violated time reversal invariance *Phys. Rev. Lett.* **117** 144101
[132] Poli C, Luna-Acosta G A and Stöckmann H-J 2012 Nearest level spacing statistics in open chaotic systems: Generalization of the Wigner surmise *Phys. Rev. Lett.* **108** 174101
[133] Richens P and Berry M 1981 Pseudo-integrable systems in classical and quantum mechanics *Physica* D **2** 495–512
[134] Berry M V and Robnik M 1984 Semiclassical level spacings when regular and chaotic orbits coexist *J. Phys. A: Math. Gen.* **17** 2413–21
[135] Cheon T and Cohen T D 1989 Quantum level statistics of pseudo-integrable billiards *Phys. Rev. Lett.* **62** 2769–72
[136] Izrailev F M 1989 Intermediate statistics of the quasi-energy spectrum and quantum localisation of classical chaos *J. Phys. A: Math. Gen.* **22** 865–78
[137] Šeba P 1990 Wave chaos in singular quantum billiard *Phys. Rev. Lett.* **64** 1855–8
[138] Prosen T and Robnik M 1994 Semiclassical energy level statistics in the transition region between integrability and chaos: transition from Brody-like to Berry–Robnik behaviour *J. Phys. A: Math. Gen.* **27** 8059–77
[139] Bogomolny E B, Gerland U and Schmit C 1999 Models of intermediate spectral statistics *Phys. Rev.* E **59** R1315–8
[140] Shigehara T, Yoshinaga N, Cheon T and Mizusaki T 1993 Level-spacing distribution of a singular billiard *Phys. Rev.* E **47** R3822–5
[141] Shigehara T 1994 Conditions for the appearance of wave chaos in quantum singular systems with a pointlike scatterer *Phys. Rev.* E **50** 4357–70
[142] Bogomolny E, Gerland U and Schmit C 2001 Singular statistics *Phys. Rev.* E **63** 036206
[143] Bogomolny E, Giraud O and Schmit C 2002 Nearest-neighbor distribution for singular billiards *Phys. Rev.* E **65** 056214
[144] García-García A M and Wang J 2006 Semi-Poisson statistics in quantum chaos *Phys. Rev.* E **73** 036210
[145] Tuan P H, Liang H C, Tung J C, Chiang P Y, Huang K F and Chen Y F 2015 Manifesting the evolution of eigenstates from quantum billiards to singular billiards in the strongly coupled limit with a truncated basis by using RLC networks *Phys. Rev.* E **92** 062906

[146] Białous M, Yunko V, Bauch S, Ławniczak M, Dietz B and Sirko L 2016 Long-range correlations in rectangular cavities containing pointlike perturbations *Phys. Rev.* E **94** 042211
[147] Shklovskii B I, Shapiro B, Sears B R, Lambrianides P and Shore H B 1993 Statistics of spectra of disordered systems near the metal-insulator transition *Phys. Rev.* B **47** 11487–90
[148] Shapiro M, Ronkin J and Brumer P 1988 Scaling laws and correlation lengths of quantum and classical ergodic states *Chem. Phys. Lett.* **148** 177–82
[149] Srivasta N and Muller G 1990 Quantum images of Hamiltonian chaos *Phys. Lett.* A **147** 282–6
[150] Lan B L and Wardlaw D M 1993 Signatures of chaos in the modulus and phase of time-dependent wave functions *Phys. Rev.* E **47** 2176–9
[151] Wang W, Xu G and Fu D 1994 Manifestation of destruction of quantum integrability with expectation and uncertainty values of quantum observables *Phys. Lett.* A **190** 377–81
[152] Bies W E, Kaplan L, Haggerty M R and Heller E J 2001 Localization of eigenfunctions in the stadium billiard *Phys. Rev.* E **63** 066214
[153] Blum G, Gnutzmann S and Smilansky U 2002 Nodal domains statistics: a criterion for quantum chaos *Phys. Rev. Lett.* **88** 114101
[154] Berkolaiko G, Keating J P and Winn B 2003 Intermediate wave function statistics *Phys. Rev. Lett.* **91** 134103
[155] Kotimäki V, Räsänen E, Hennig H and Heller E J 2013 Fractal dynamics in chaotic quantum transport *Phys. Rev.* E **88** 022913
[156] Mason D J, Borunda M F and Heller E J 2015 Revealing the flux: using processed Husimi maps to visualize dynamics of bound systems and mesoscopic transport *Phys. Rev.* B **91** 165405
[157] Jain S R and Samajdar R 2017 Nodal portraits of quantum billiards: domains, lines, and statistics *Rev. Mod. Phys.* **89** 045005
[158] Blümel R and Smilansky U 1988 Classical irregular scattering and its quantum-mechanical implications *Phys. Rev. Lett.* **60** 477–80
[159] Lewenkopf C and Weidenm H 1991 Stochastic versus semiclassical approach to quantum chaotic scattering *Ann. Phys.* **212** 53–83
[160] Lai Y-C, Blümel R, Ott E and Grebogi C 1992 Quantum manifestations of chaotic scattering *Phys. Rev. Lett.* **68** 3491–4
[161] Baranger H U, Jalabert R A and Stone A D 1993 Quantum chaotic scattering effects in semiconductor microstructures *Chaos* **3** 665–82
[162] Ketzmerick R 1996 Fractal conductance fluctuations in generic chaotic cavities *Phys. Rev.* B **54** 10841–4
[163] Huckestein B, Ketzmerick R and Lewenkopf C H 2000 Quantum transport through ballistic cavities: soft vs hard quantum chaos *Phys. Rev. Lett.* **84** 5504–7
[164] Casati G, Guarneri I and Maspero G 2000 Fractal survival probability fluctuations *Phys. Rev. Lett.* **84** 63–6
[165] Bäcker A, Manze A, Huckestein B and Ketzmerick R 2002 Isolated resonances in conductance fluctuations and hierarchical states *Phys. Rev.* E **66** 016211
[166] Huang W-M, Mou C-Y and Chang C-H 2010 Scattering phase correction for semiclassical quantization rules in multi-dimensional quantum systems *Commun. Theor. Phys.* **53** 250
[167] Marcus C M, Rimberg A J, Westervelt R M, Hopkins P F and Gossard A C 1992 Conductance fluctuations and chaotic scattering in ballistic microstructures *Phys. Rev. Lett.* **69** 506–9
[168] Marcus C M, Westervelt R M, Hopkins P F and Gossard A C 1993 Conductance fluctuations and quantum chaotic scattering in semiconductor microstructures *Chaos* **3** 643–53
[169] Taylor R P *et al* 1997 Self-similar magnetoresistance of a semiconductor Sinai billiard *Phys. Rev. Lett.* **78** 1952–5
[170] Sachrajda A S, Ketzmerick R, Gould C, Feng Y, Kelly P J, Delage A and Wasilewski Z 1998 Fractal conductance fluctuations in a soft-wall stadium and a Sinai billiard *Phys. Rev. Lett.* **80** 1948–51
[171] Blanchard P and Olkiewicz R 2003 Decoherence-induced continuous pointer states *Phys. Rev. Lett.* **90** 010403
[172] Ferry D K, Akis R and Bird J P 2004 Einselection in action: Decoherence and pointer states in open quantum dots *Phys. Rev. Lett.* **93** 026803
[173] Ferry D K, Huang L, Yang R, Lai Y-C and Akis R 2010 Open quantum dots in graphene: scaling relativistic pointer states *J. Phys.: Conf. Ser.* **220** 012015
[174] Wang W-G, He L and Gong J 2012 Preferred states of decoherence under intermediate system-environment coupling *Phys. Rev. Lett.* **108** 070403
[175] Yang Y-B and Wang W-G 2015 A phenomenon of decoherence induced by chaotic environment, Chinese *Phys. Lett.* **32** 030301
[176] Lee P A and Stone A D 1985 universal conductance fluctuations in metals *Phys. Rev. Lett.* **55** 1622–5
[177] Kaplan S B and Hartstein A 1986 universal conductance fluctuations in narrow Si accumulation layers *Phys. Rev. Lett.* **56** 2403–6
[178] Skocpol W J, Mankiewich P M, Howard R E, Jackel L D, Tennant D M and Stone A D 1986 universal conductance fluctuations in silicon inversion-layer nanostructures *Phys. Rev. Lett.* **56** 2865–8
[179] Iida S, Weidenmüller H A and Zuk J A 1990 Wave propagation through disordered media and universal conductance fluctuations *Phys. Rev. Lett.* **64** 583–6
[180] Kharitonov M Y and Efetov K B 2008 Universal conductance fluctuations in graphene *Phys. Rev.* B **78** 033404
[181] Tomsovic S and Ullmo D 1994 Chaos-assisted tunneling *Phys. Rev.* E **50** 145–62
[182] Dembowski C, Gräf H-D, Heine A, Hofferbert R, Rehfeld H and Richter A 2000 First experimental evidence for chaos-assisted tunneling in a microwave annular billiard *Phys. Rev. Lett.* **84** 867–70
[183] Steck D A, Oskay W H and Raizen M G 2001 Observation of chaos-assisted tunneling between islands of stability *Science* **293** 274–8
[184] Tomsovic S 2001 Tunneling and chaos *Phys. Scr.* **T90** 162–5
[185] de Moura A P S, Lai Y-C, Akis R, Bird J and Ferry D K 2002 Tunneling and nonhyperbolicity in quantum dots *Phys. Rev. Lett.* **88** 236804
[186] Bäcker A, Ketzmerick R and Monastra A G 2005 Flooding of chaotic eigenstates into regular phase space islands *Phys. Rev. Lett.* **94** 054102
[187] Bäcker A, Ketzmerick R, Löck S, Robnik M, Vidmar G, Höhmann R, Kuhl U and Stöckmann H-J 2008 Dynamical tunneling in mushroom billiards *Phys. Rev. Lett.* **100** 174103
[188] Bäcker A, Ketzmerick R, Löck S and Schilling L 2008 Regular-to-chaotic tunneling rates using a fictitious integrable system *Phys. Rev. Lett.* **100** 104101
[189] Rong S, Hai W, Xie Q and Zhu Q 2009 Chaos enhancing tunneling in a coupled Bose–Einstein condensate with a double driving *Chaos* **19** 033129
[190] Löck S, Bäcker A, Ketzmerick R and Schlagheck P 2010 Regular-to-chaotic tunneling rates: from the quantum to the semiclassical regime *Phys. Rev. Lett.* **104** 114101
[191] Pecora L M, Lee H, Wu D H, Antonsen T, Lee M J and Ott E 2011 Chaos regularization of quantum tunneling rates *Phys. Rev.* E **83** 065201
[192] Lee M J, Antonsen T M, Ott E and Pecora L M 2012 Theory of chaos regularization of tunneling in chaotic quantum dots *Phys. Rev.* E **86** 056212

[193] Ni X, Huang L, Lai Y-C and Pecora L M 2012 Effect of chaos on relativistic quantum tunneling *Europhys. Lett.* **98** 50007
[194] Ni X, Huang L, Ying L and Lai Y-C 2013 Relativistic quantum tunneling of a Dirac fermion in nonhyperbolic chaotic systems *Phys. Rev.* B **87** 224304
[195] Ying L, Wang G, Huang L and Lai Y-C 2014 Quantum chaotic tunneling in graphene systems with electron-electron interactions *Phys. Rev.* B **90** 224301
[196] Ugajin R 1997 Spectral statistics of correlated electrons in a square-well quantum dot *Physica* A **237** 220–8
[197] Zhang C, Liu J, Raizen M G and Niu Q 2004 Quantum chaos of Bogoliubov waves for a Bose–Einstein condensate in stadium billiards *Phys. Rev. Lett.* **93** 074101
[198] Hagymási I, Vancsó P, Pálinkás A and Osváth Z 2017 Interaction effects in a chaotic graphene quantum billiard *Phys. Rev.* B **95** 075123
[199] Bychek A A, Muraev P S and Kolovsky A R 2019 Probing quantum chaos in many-body quantum systems by the induced dissipation *Phys. Rev.* A **100** 013610
[200] Peres A 1984 Stability of quantum motion in chaotic and regular systems *Phys. Rev.* A **30** 1610–5
[201] Bonci L, Roncaglia R, West B J and Grigolini P 1991 Quantum irreversibility and chaos *Phys. Rev. Lett.* **67** 2593–6
[202] Tomsovic S and Heller E J 1991 Semiclassical dynamics of chaotic motion: Unexpected long-time accuracy *Phys. Rev. Lett.* **67** 664–7
[203] Tomsovic S and Heller E J 1993 Long-time semiclassical dynamics of chaos: the stadium billiard *Phys. Rev.* E **47** 282–99
[204] Jie Q and Xu G 1995 Quantum signature of classical chaos in the temporal mean of expectation values of observables *Chin. Phys. Lett.* **12** 577–80
[205] Xie R-H and Xu G-O 1996 Quantum signature of classical chaos in a Lipkin model: Sensitivity of eigenfunctions to parameter perturbations *Chin. Phys. Lett.* **13** 329–32
[206] Jie Q-L and Xu G-O 1996 Numerical evidence of quantum correspondence to the classical ergodicity *Commun. Theor. Phys.* **26** 191–6
[207] Xing Y-Z, Xu G-O and Li J-Q 2001 The relation between one-to-one correspondent orthonormal eigenstates of h_0 and $h(\lambda) = h_0 + \lambda v$ *Commun. Theor. Phys.* **35** 11–4
[208] Jacquod P, Silvestrov P and Beenakker C 2001 Golden rule decay versus Lyapunov decay of the quantum Loschmidt echo *Phys. Rev.* E **64** 055203
[209] Jalabert R A and Pastawski H M 2001 Environment-independent decoherence rate in classically chaotic systems *Phys. Rev. Lett.* **86** 2490–3
[210] Li J-Q, Liu F, Xing Y-Z and Zuo W 2002 Quantitative measurement of the exponential growth of spreading width of a quantum wave packet in chaotic systems *Commun. Theor. Phys.* **37** 671–4
[211] Karkuszewski Z P, Jarzynski C and Zurek W H 2002 Quantum chaotic environments, the butterfly effect, and decoherence *Phys. Rev. Lett.* **89** 170405
[212] Cerruti N R and Tomsovic S 2002 Sensitivity of wave field evolution and manifold stability in chaotic systems *Phys. Rev. Lett.* **88** 054103
[213] Cucchietti F M, Lewenkopf C H, Mucciolo E R, Pastawski H M and Vallejos R O 2002 Measuring the Lyapunov exponent using quantum mechanics *Phys. Rev.* E **65** 046209
[214] Cucchietti F M, Dalvit D A R, Paz J P and Zurek W H 2003 Decoherence and the Loschmidt echo *Phys. Rev. Lett.* **91** 210403
[215] Jacquod P, Adagideli İ and Beenakker C W J 2003 Anomalous power law of quantum reversibility for classically regular dynamics *Europhys. Lett.* **61** 729–35
[216] Adamov Y, Gornyi I V and Mirlin A D 2003 Loschmidt echo and Lyapunov exponent in a quantum disordered system *Phys. Rev.* E **67** 056217
[217] Vaníček J and Heller E J 2003 Semiclassical evaluation of quantum fidelity *Phys. Rev.* E **68** 056208
[218] Sankaranarayanan R and Lakshminarayan A 2003 Recurrence of fidelity in nearly integrable systems *Phys. Rev.* E **68** 036216
[219] Gorin T, Prosen T, Seligman T H and Strunz W T 2004 Connection between decoherence and fidelity decay in echo dynamics *Phys. Rev.* A **70** 042105
[220] Liu J, Wang W, Zhang C, Niu Q and Li B 2005 Fidelity for the quantum evolution of a Bose–Einstein condensate *Phys. Rev.* A **72** 063623
[221] Weinstein Y S and Hellberg C S 2005 Quantum fidelity decay in quasi-integrable systems *Phys. Rev.* E **71** 016209
[222] Prosen T and Žnidarič M 2005 Quantum freeze of fidelity decay for chaotic dynamics *Phys. Rev. Lett.* **94** 044101
[223] Wang W-G and Li B 2005 Uniform semiclassical approach to fidelity decay: from weak to strong perturbation *Phys. Rev.* E **71** 066203
[224] Gorin T, Prosen T, Seligman T H and Z˘nidaric M 2006 Dynamics of Loschmidt echoes and fidelity decay *Phys. Rep.* **435** 33–156
[225] Quan H T, Song Z, Liu X F, Zanardi P and Sun C P 2006 Decay of Loschmidt echo enhanced by quantum criticality *Phys. Rev. Lett.* **96** 140604
[226] Pellegrini F and Montangero S 2007 Fractal fidelity as a signature of quantum chaos *Phys. Rev.* A **76** 052327
[227] Zanardi P, Quan H T, Wang X and Sun C P 2007 Mixed-state fidelity and quantum criticality at finite temperature *Phys. Rev.* A **75** 032109
[228] Ian H, Gong Z R, Liu Y-X, Sun C P and Nori F 2008 Cavity optomechanical coupling assisted by an atomic gas *Phys. Rev.* A **78** 013824
[229] Höhmann R, Kuhl U and Stöckmann H-J 2008 Algebraic fidelity decay for local perturbations *Phys. Rev. Lett.* **100** 124101
[230] Huang J-F, Li Y, Liao J-Q, Kuang L-M and Sun C P 2009 Dynamic sensitivity of photon-dressed atomic ensemble with quantum criticality *Phys. Rev.* A **80** 063829
[231] Quan H T and Cucchietti F M 2009 Quantum fidelity and thermal phase transitions *Phys. Rev.* E **79** 031101
[232] Gutkin B, Waltner D, Gutiérrez M, Kuipers J and Richter K 2010 Quantum corrections to fidelity decay in chaotic systems *Phys. Rev.* E **81** 036222
[233] Kohler H, Sommers H-J, Åberg S and Guhr T 2010 Exact fidelity and full fidelity statistics in regular and chaotic surroundings *Phys. Rev.* E **81** 050103
[234] Rams M M and Damski B 2011 Quantum fidelity in the thermodynamic limit *Phys. Rev. Lett.* **106** 055701
[235] Cai C Y, Ai Q, Quan H T and Sun C P 2012 Sensitive chemical compass assisted by quantum criticality *Phys. Rev.* A **85** 022315
[236] Wang W-G 2019 A renormalized-Hamiltonian-flow approach to eigenenergies and eigenfunctions *Commun. Theor. Phys.* **71** 861
[237] Stöckmann H J and Stein J 1990 Quantum chaos in billiards studied by microwave-absorption *Phys. Rev. Lett.* **64** 2215–8
[238] Doron E, Smilansky U and Frenkel A 1990 Experimental demonstration of chaotic scattering of microwaves *Phys. Rev. Lett.* **65** 3072–5
[239] Haake F, Lenz G, Seba P, Stein J, Stöckmann H-J and Życzkowski K 1991 Manifestation of wave chaos in pseudo-integrable microwave resonators *Phys. Rev.* A **44** R6161–4
[240] Gräf H-D, Harney H L, Lengeler H, Lewenkopf C H, Rangacharyulu C, Richter A, Schardt P and Weidenmüller H A 1992 Distribution of eigenmodes in a superconducting stadium billiard with chaotic dynamics *Phys. Rev. Lett.* **69** 1296–9

[241] Kudrolli A, Sridhar S, Pandey A and Ramaswamy R 1994 Signatures of chaos in quantum billiards: microwave experiments *Phys. Rev.* E **49** R11–4
[242] So P, Anlage S M, Ott E and Oerter R N 1995 Wave chaos experiments with and without time reversal symmetry: GUE and GOE statistics *Phys. Rev. Lett.* **74** 2662–5
[243] Deus S, Koch P M and Sirko L 1995 Statistical properties of the eigenfrequency distribution of three-dimensional microwave cavities *Phys. Rev.* E **52** 1146–55
[244] Stoffregen U, Stein J, Stöckmann H-J, Kuś M and Haake F 1995 Microwave billiards with broken time reversal symmetry *Phys. Rev. Lett.* **74** 2666–9
[245] Alt H, Gräf H-D, Hofferbert R, Rangacharyulu C, Rehfeld H, Richter A, Schardt P and Wirzba A 1996 Chaotic dynamics in a three-dimensional superconducting microwave billiard *Phys. Rev.* E **54** 2303–12
[246] Alt H, Gräf H-D, Guhr T, Harney H L, Hofferbert R, Rehfeld H, Richter A and Schardt P 1997 Correlation-hole method for the spectra of superconducting microwave billiards *Phys. Rev.* E **55** 6674–83
[247] Sirko L, Koch P M and Blümel R 1997 Experimental identification of non-newtonian orbits produced by ray splitting in a dielectric-loaded microwave cavity *Phys. Rev. Lett.* **78** 2940–3
[248] Kottos T, Smilansky U, Fortuny J and Nesti G 1999 Chaotic scattering of microwaves *Radio Sci.* **34** 747–58
[249] Sirko L, Bauch S, Hlushchuk Y, Koch P, Blümel R, Barth M, Kuhl U and Stöckmann H-J 2000 Observation of dynamical localization in a rough microwave cavity *Phys. Lett.* A **266** 331–5
[250] Stöckmann H-J, Barth M, Dörr U, Kuhl U and Schanze H 2001 Microwave studies of chaotic billiards and disordered systems *Physica* E **9** 571–7
[251] Dembowski C, Dietz B, Gräf H-D, Heine A, Leyvraz F, Miski-Oglu M, Richter A and Seligman T H 2003 Phase shift experiments identifying Kramers doublets in a chaotic superconducting microwave billiard of threefold symmetry *Phys. Rev. Lett.* **90** 014102
[252] Kuhl U, Stöckmann H-J and Weaver R 2005 Classical wave experiments on chaotic scattering *J. Phys. A: Math. Gen.* **38** 10433–63
[253] Dietz B and Richter A 2015 Quantum and wave dynamical chaos in superconducting microwave billiards *Chaos* **25** 097601
[254] Zhang R, Zhang W, Dietz B, Chai G and Huang L 2019 Experimental investigation of the fluctuations in nonchaotic scattering in microwave billiards *Chin. Phys.* B **28** 100502
[255] Slusher R and Weisbuch C 1994 Optical microcavities in condensed matter systems *Solid State Commun.* **92** 149–58
[256] Gmachl C, Capasso F, Narimanov E E, Nockel J U, Stone A D, Faist J, Sivco D L and Cho A Y 1998 High-power directional emission from microlasers with chaotic resonators *Science* **280** 1556–64
[257] Vahala K J 2003 Optical microcavities *Nature* **424** 839–46
[258] Lee S-B, Yang J, Moon S, Lee S-Y, Shim J-B, Kim S W, Lee J-H and An K 2009 Observation of an exceptional point in a chaotic optical microcavity *Phys. Rev. Lett.* **103** 134101
[259] Song Q, Fang W, Liu B, Ho S-T, Solomon G S and Cao H 2009 Chaotic microcavity laser with high quality factor and unidirectional output *Phys. Rev.* A **80** 041807
[260] Peng B, Oezdemir S K, Lei F, Monifi F, Gianfreda M, Long G L, Fan S, Nori F, Bender C M and Yang L 2014 Parity-time-symmetric whispering-gallery microcavities *Nat. Phys.* **10** 394–8
[261] Wang L, Lippolis D, Li Z, Jiang X, Gong Q and Xiao Y 2016 Statistics of chaotic resonances in an optical microcavity *Phys. Rev.* E **93** 040201
[262] Jiang X, Shao L, Zhang S, Yi X, Wiersig J, Wang L, Gong Q, Loncar M, Yang L and Xiao Y 2017 Chaos-assisted broadband momentum transformation in optical microresonators *Science* **358** 344–7
[263] Bittner S, Guazzotti S, Zeng Y, Hu X, Yilmaz H, Kim K, Oh S S, Wang Q J, Hess O and Cao H 2018 Suppressing spatiotemporal lasing instabilities with wave-chaotic microcavities *Science* **361** 1225–31
[264] Guidry M A *et al* 2019 Three-dimensional micro-billiard lasers: the square pyramid *Europhys. Lett.* **126** 64004
[265] Legrand O, Schmit C and Sornette D 1992 Quantum chaos methods applied to high-frequency plate vibrations *Europhys. Lett.* **18** 101–6
[266] Mortessagne F, Legrand O and Sornette D 1993 Transient chaos in room acoustics *Chaos* **3** 529–41
[267] Ellegaard C, Guhr T, Lindemann K, Lorensen H Q, Nygård J and Oxborrow M 1995 Spectral statistics of acoustic resonances in aluminum blocks *Phys. Rev. Lett.* **75** 1546–9
[268] Ellegaard C, Guhr T, Lindemann K, Nygård J and Oxborrow M 1996 Symmetry breaking and spectral statistics of acoustic resonances in quartz blocks *Phys. Rev. Lett.* **77** 4918–21
[269] Leitner D M 1997 Effects of symmetry breaking on statistical properties of near-lying acoustic resonances *Phys. Rev.* E **56** 4890–1
[270] Bogomolny E and Hugues E 1998 Semiclassical theory of flexural vibrations of plates *Phys. Rev.* E **57** 5404–24
[271] Lindelof P E, Norregaard J and Hanberg J 1986 New light on the scattering mechanisms in Si inversion layers by weak localization experiments *Phys. Scr.* **1986** 17–26
[272] Nunez-Fernandez Y, Trallero-Giner C and Buchleitner A 2008 Liquid surface waves in parabolic tanks *Phys. Fluids* **20** 117106
[273] Tang Y, Shen Y, Yang J, Liu X, Zi J and Li B 2008 Experimental evidence of wave chaos from a double slit experiment with water surface waves *Phys. Rev.* E **78** 047201
[274] Casati G, Chirikov B V, Izraelev F M and Ford J 1979 Stochastic behavior of a quantum pendulum under a periodic perturbation *Stochastic Behavior in Classical and Quantum Hamiltonian Systems* ed G Casati and J Ford (Berlin: Springer) pp 334–52
[275] Fishman S, Grempel D R and Prange R E 1982 Chaos, quantum recurrences, and Anderson localization *Phys. Rev. Lett.* **49** 509–12
[276] Grempel D R, Fishman S and Prange R E 1982 Localization in an incommensurate potential: an exactly solvable model *Phys. Rev. Lett.* **49** 833–6
[277] Prange R E, Grempel D R and Fishman S 1984 Solvable model of quantum motion in an incommensurate potential *Phys. Rev.* B **29** 6500–12
[278] Berry M V 1984 Incommensurability in an exactly-soluble quantal and classical model for a kicked rotator *Physica* D **10** 369–78
[279] Fishman S, Prange R E and Griniasty M 1989 Scaling theory for the localization length of the kicked rotor *Phys. Rev.* A **39** 1628–33
[280] Izrailev F M 1990 Simple models of quantum chaos: spectrum and eigenfunctions *Phys. Rep.* **196** 299–392
[281] Chirikov B V 1991 A theory of quantum diffusion localization *Chaos* **1** 95–100
[282] Iomin A, Fishman S and Zaslavsky G M 2002 Quantum localization for a kicked rotor with accelerator mode islands *Phys. Rev.* E **65** 036215
[283] Gong J and Wang J 2007 Quantum diffusion dynamics in nonlinear systems: a modified kicked-rotor model *Phys. Rev.* E **76** 036217
[284] García-García A M and Wang J 2008 Universality in quantum chaos and the one-parameter scaling theory *Phys. Rev. Lett.* **100** 070603

[285] Zhao W-L and Jie Q-L 2009 Quantum to classical transition in a system of two coupled kicked rotors *Commun. Theor. Phys.* **51** 465

[286] Wang H, Wang J, Guarneri I, Casati G and Gong J 2013 Exponential quantum spreading in a class of kicked rotor systems near high-order resonances *Phys. Rev.* E **88** 052919

[287] Wang J, Tian C and Altland A 2014 Unconventional quantum criticality in the kicked rotor *Phys. Rev.* B **89** 195105

[288] Fang P and Wang J 2016 Superballistic wavepacket spreading in double kicked rotors *Sci. China-Phys. Mech. Astron.* **59** 680011

[289] Wang X, Ghose S, Sanders B C and Hu B 2004 Entanglement as a signature of quantum chaos *Phys. Rev.* E **70** 016217

[290] Ghose S, Stock R, Jessen P, Lal R and Silberfarb A 2008 Chaos, entanglement, and decoherence in the quantum kicked top *Phys. Rev.* A **78** 042318

[291] Lombardi M and Matzkin A 2011 Entanglement and chaos in the kicked top *Phys. Rev.* E **83** 016207

[292] Prange R E and Fishman S 1989 Experimental realizations of kicked quantum chaotic systems *Phys. Rev. Lett.* **63** 704–7

[293] Ammann H, Gray R, Shvarchuck I and Christensen N 1998 Quantum delta-kicked rotor: experimental observation of decoherence *Phys. Rev. Lett.* **80** 4111–5

[294] Chaudhury S, Smith A, Anderson B, Ghose S and Jessen P S 2009 Quantum signatures of chaos in a kicked top *Nature* **461** 768

[295] Hainaut C, Fang P, Rançon A, Clément J-F M C, Szriftgiser P, Garreau J-C, Tian C and Chicireanu R 2018 Experimental observation of a time-driven phase transition in quantum chaos *Phys. Rev. Lett.* **121** 134101

[296] Wang J, Monteiro T S, Fishman S, Keating J P and Schubert R 2007 Fractional $\hbar$ scaling for quantum kicked rotors without cantori *Phys. Rev. Lett.* **99** 234101

[297] Wang J and García-García A M 2009 Anderson transition in a three-dimensional kicked rotor *Phys. Rev.* E **79** 036206

[298] Wang J and Gong J 2009 Butterfly floquet spectrum in driven SU(2) systems *Phys. Rev. Lett.* **102** 244102

[299] Zhou L and Gong J 2018 Floquet topological phases in a spin-$1/2$ double kicked rotor *Phys. Rev.* A **97** 063603

[300] Dicke R H 1954 Coherence in spontaneous radiation processes *Phys. Rev.* **93** 99–110

[301] Lewenkopf C H, Nemes M C, Marvulle V, Pato M P and Wreszinski W F 1991 Level statistics transitions in the spin-boson model *Phys. Lett.* A **155** 113–6

[302] Cibils M B, Cuche Y, Marvulle V, Wreszinski W F, Amiet J P and Beck H 1991 The semiclassical limit of the spin boson model *J. Phys. A: Math. Gen.* **24** 1661–75

[303] Emary C and Brandes T 2003 Quantum chaos triggered by precursors of a quantum phase transition: the Dicke model *Phys. Rev. Lett.* **90** 044101

[304] Altland A and Haake F 2012 Quantum chaos and effective thermalization *Phys. Rev. Lett.* **108** 073601

[305] Chávez-Carlos J, López-del Carpio B, Bastarrachea-Magnani M A, Stránský P, Lerma-Hernández S, Santos L F and Hirsch J G 2019 Quantum and classical Lyapunov exponents in atom-field interaction systems *Phys. Rev. Lett.* **122** 024101

[306] Lewisswan R J, Safavinaini A, Bollinger J J and Rey A M 2019 Unifying scrambling, thermalization and entanglement through measurement of fidelity out-of-time-order correlators in the Dicke model *Nat. Commun.* **10** 1581

[307] Lipkin H J, Meshkov N and Glick A J 1965 Validity of many-body approximation methods for a solvable model. i. exact solutions and perturbation theory *Nucl. Phys.* **62** 188–98

[308] Meredith D C, Koonin S E and Zirnbauer M R 1988 Quantum chaos in a schematic shell model *Phys. Rev.* A **37** 3499–513

[309] Relaño A 2018 Anomalous thermalization in quantum collective models *Phys. Rev. Lett.* **121** 030602

[310] Wang J and Wang W-G 2018 Characterization of random features of chaotic eigenfunctions in unperturbed basis *Phys. Rev.* E **97** 062219

[311] Geisel T, Radons G and Rubner J 1986 Kolmogorov-Arnol'd-Moser barriers in the quantum dynamics of chaotic systems *Phys. Rev. Lett.* **57** 2883–6

[312] Fishman S, Grempel D R and Prange R E 1987 Temporal crossover from classical to quantal behavior near dynamical critical points *Phys. Rev.* A **36** 289–305

[313] Radons G and Prange R E 1988 Wave functions at the critical Kolmogorov-Arnol'd-Moser surface *Phys. Rev. Lett.* **61** 1691–4

[314] Maitra N T and Heller E J 2000 Quantum transport through cantori *Phys. Rev.* E **61** 3620–31

[315] Creffield C E, Hur G and Monteiro T S 2006 Localization-delocalization transition in a system of quantum kicked rotors *Phys. Rev. Lett.* **96** 024103

[316] Creffield C E, Fishman S and Monteiro T S 2006 Theory of 2δ-kicked quantum rotors *Phys. Rev.* E **73** 066202

[317] Hufnagel L, Ketzmerick R, Kottos T and Geisel T 2001 Superballistic spreading of wave packets *Phys. Rev.* E **64** 012301

[318] Zhang Z, Tong P, Gong J and Li B 2012 Quantum hyperdiffusion in one-dimensional tight-binding lattices *Phys. Rev. Lett.* **108** 070603

[319] Qin P, Yin C and Chen S 2014 Dynamical Anderson transition in one-dimensional periodically kicked incommensurate lattices *Phys. Rev.* B **90** 054303

[320] Likhachev V N, Shevaleevskii O I and Vinogradov G A 2016 Quantum dynamics of charge transfer on the one-dimensional lattice: wave packet spreading and recurrence *Chin. Phys.* B **25** 018708

[321] Zhao W-L, Gong J, Wang W-G, Casati G, Liu J and Fu L-B 2016 Exponential wave-packet spreading via self-interaction time modulation *Phys. Rev.* A **94** 053631

[322] Gholami E and Lashkami Z M 2017 Noise, delocalization, and quantum diffusion in one-dimensional tight-binding models *Phys. Rev.* E **95** 022216

[323] Bayfield J E and Koch P M 1974 Multiphoton ionization of highly excited hydrogen atoms *Phys. Rev. Lett.* **33** 258–61

[324] Leopold J G and Percival I C 1978 Microwave ionization and excitation of Rydberg atoms *Phys. Rev. Lett.* **41** 944–7

[325] Casati G, Chirikov B V and Shepelyansky D L 1984 Quantum limitations for chaotic excitation of the hydrogen atom in a monochromatic field *Phys. Rev. Lett.* **53** 2525–8

[326] van Leeuwen K A H, Oppen G v, Renwick S, Bowlin J B, Koch P M, Jensen R V, Rath O, Richards D and Leopold J G 1985 Microwave ionization of hydrogen atoms: experiment versus classical dynamics *Phys. Rev. Lett.* **55** 2231–4

[327] Casati G, Chirikov B V, Guarneri I and Shepelyansky D L 1986 Dynamical stability of quantum 'chaotic' motion in a hydrogen atom *Phys. Rev. Lett.* **56** 2437–40

[328] Casati G, Chirikov B V, Shepelyansky D L and Guarneri I 1986 New photoelectric ionization peak in the hydrogen atom *Phys. Rev. Lett.* **57** 823–6

[329] Casati G, Chirikov B V, Guarneri I and Shepelyansky D L 1987 Localization of diffusive excitation in the two-dimensional hydrogen atom in a monochromatic field *Phys. Rev. Lett.* **59** 2927–30

[330] Casati G, Chirikov B V, Shepelyansky D L and Guarneri I 1987 Relevance of classical chaos in quantum mechanics: the hydrogen atom in a monochromatic field *Phys. Rep.* **154** 77–123

[331] Casati G, Chirikov B V, Shepelyansky D L and Guarneri I 1987 Relevance of classical chaos in quantum mechanics: The hydrogen atom in a monochromatic field *Phys. Rep.* **154** 77–123

[332] Chirikov B V, Izrailev F M and Shepelyansky D L 1988 Quantum chaos: localization versus ergodicity *Physica* D **33** 77–88
[333] Blümel R and Smilansky U 1989 Ionization of excited hydrogen atoms by microwave fields: a test case for quantum chaos *Phys. Scr.* **40** 386–93
[334] Bayfield J E, Casati G, Guarneri I and Sokol D W 1989 Localization of classically chaotic diffusion for hydrogen atoms in microwave fields *Phys. Rev. Lett.* **63** 364–7
[335] Blümel R, Graham R, Sirko L, Smilansky U, Walther H and Yamada K 1989 Microwave excitation of Rydberg atoms in the presence of noise *Phys. Rev. Lett.* **62** 341–4
[336] Jensen R, Susskind S and Sanders M 1991 Chaotic ionization of highly excited hydrogen atoms: Comparison of classical and quantum theory with experiment *Phys. Rep.* **201** 1–56
[337] Yoakum S, Sirko L and Koch P M 1992 Stueckelberg oscillations in the multiphoton excitation of helium Rydberg atoms: Observation with a pulse of coherent field and suppression by additive noise *Phys. Rev. Lett.* **69** 1919–22
[338] Haffmans A, Blümel R, Koch P M and Sirko L 1994 Prediction of a new peak in two-frequency microwave 'ionization' of excited hydrogen atoms *Phys. Rev. Lett.* **73** 248–51
[339] Koch P M and van Leeuwen K H A 1995 The importance of resonances in microwave 'ionization' of excited hydrogen atoms *Phys. Rep.* **256** 289–403
[340] Kang S and Chen C-Y 2010 Statistics and correlation properties of diamagnetic high Rydberg hydrogen atom *Commun. Theor. Phys.* **53** 105
[341] Xu X, Zhang Y, Cai X, Zhao G and Kang L 2016 Fractal dynamics in the ionization of helium Rydberg atoms *Chin. Phys.* B **25** 110301
[342] Aßmann M, Thewes J, Fröhlich D and Bayer M 2016 Quantum chaos and breaking of all anti-unitary symmetries in Rydberg excitons *Nat. Mater.* **15** 741–5
[343] Zhang Y, Xu X, Kang L, Cai X and Tang X 2018 Analysis of the fractal intrinsic quality in the ionization of Rydberg helium and lithium atoms *Chin. Phys.* B **27** 053401
[344] Xu L and Fu L-B 2019 Understanding tunneling ionization of atoms in laser fields using the principle of multiphoton absorption *Chin.. Phys. Lett.* **36** 043202
[345] Salmond G L, Holmes C A and Milburn G J 2002 Dynamics of a strongly driven two-component Bose–Einstein condensate *Phys. Rev.* A **65** 033623
[346] Gardiner S A 2002 Quantum chaos in Bose–Einstein condensates *J. Mod. Opt.* **49** 1971–7
[347] Franzosi R and Penna V 2003 Chaotic behavior, collective modes, and self-trapping in the dynamics of three coupled Bose–Einstein condensates *Phys. Rev.* E **67** 046227
[348] Xie Q and Hai W 2005 Quantum entanglement and chaos in kicked two-component Bose–Einstein condensates *Euro. Phys. J.* D **33** 265–72
[349] Mahmud K W, Perry H and Reinhardt W P 2005 Quantum phase-space picture of Bose–Einstein condensates in a double well *Phys. Rev.* A **71** 023615
[350] Jun X, Wen-Hua H and Hui L 2007 Generation and control of chaos in a Bose–Einstein condensate *Chin. Phys.* **16** 2244
[351] Kronjäger J, Sengstock K and Bongs K 2008 Chaotic dynamics in spinor Bose–Einstein condensates *New J. Phys.* **10** 045028
[352] Koberle P, Cartarius H, Fabcic T, Main J and Wunner G 2009 Bifurcations, order and chaos in the Bose–Einstein condensation of dipolar gases *New J. Phys.* **11** 023017
[353] Valdez M A, Shchedrin G, Heimsoth M, Creffield C E, Sols F and Carr L D 2018 Many-body quantum chaos and entanglement in a quantum ratchet *Phys. Rev. Lett.* **120** 234101
[354] Bohigas O and Weidenmuller H 1988 Aspects of chaos in nuclear physics *Annu. Rev. Nucl. Part.* **38** 421–53
[355] Zelevinsky V and Volya A 2006 Quantum chaos and nuclear physics *Phys. Scr.* **2006** 147
[356] Raizen M G 1999 Quantum chaos with cold atoms *Adv. At. Mol. Opt. Phys.* **41** 199
[357] Klappauf B, Oskay W, Steck D and Raizen M 1999 Quantum chaos with cesium atoms: pushing the boundaries *Physica* D **131** 78–89
[358] d'Arcy M, Summy G, Fishman S and Guarneri I 2004 Novel quantum chaotic dynamics in cold atoms *Phys. Scr.* **69** C25
[359] Li H, Hai W-H and Xu J 2008 Quantum signatures of chaos in adiabatic interaction between a trapped ion and a laser standing wave *Commun. Theor. Phys.* **49** 143
[360] Krivolapov Y, Fishman S, Ott E and Antonsen T M 2011 Quantum chaos of a mixed open system of kicked cold atoms *Phys. Rev.* E **83** 016204
[361] Frisch A, Mark M, Aikawa K, Ferlaino F, Bohn J L, Makrides C, Petrov A and Kotochigova S 2014 Quantum chaos in ultracold collisions of gas-phase erbium atoms *Nature* **507** 475–9
[362] Spillane S M, Kippenberg T J and Vahala K J 2002 Ultralow-threshold Raman laser using a spherical dielectric microcavity *Nature* **415** 621–3
[363] Gensty T, Becker K, Fischer I, Elsäßer W, Degen C, Debernardi P and Bava G P 2005 Wave chaos in real-world vertical-cavity surface-emitting lasers *Phys. Rev. Lett.* **94** 233901
[364] Fang W, Cao H and Solomon G S 2007 Control of lasing in fully chaotic open microcavities by tailoring the shape factor *Appl. Phys. Lett.* **90** 081108
[365] Schack R and Caves C M 1996 Information-theoretic characterization of quantum chaos *Phys. Rev.* E **53** 3257
[366] Baranger H and Mello P 1996 Short paths and information theory in quantum chaotic scattering: transport through quantum dots *Europhys. Lett.* **33** 465
[367] Prosen T and Znidaric M 2001 Can quantum chaos enhance the stability of quantum computation? *J. Phys. A: Math. Gen.* **34** L681
[368] Poulin D, Laflamme R, Milburn G J and Paz J P 2003 Testing integrability with a single bit of quantum information *Phys. Rev.* A **68** 022302
[369] Wang X-Q, Ma J, Zhang X-H and Wang X-G 2011 Chaos and quantum fisher information in the quantum kicked top *Chin. Phys.* B **20** 050510
[370] Lashkari N, Dymarsky A and Liu H 2018 Universality of quantum information in chaotic cfts *J. High Energy Phys.* **2018** 70
[371] Pauling L 1936 The diamagnetic anisotropy of aromatic molecules *J. Chem. Phys.* **4** 673–7
[372] Kottos T and Smilansky U 1997 Quantum chaos on graphs *Phys. Rev. Lett.* **79** 4794–7
[373] Kottos T and Smilansky U 1999 Periodic orbit theory and spectral statistics for quantum graphs *Ann. Phys.* **274** 76–124
[374] Barra F and Gaspard P 2000 On the level spacing distribution in quantum graphs *J. Stat. Phys.* **101** 283–319
[375] Kottos T and Smilansky U 2000 Chaotic scattering on graphs *Phys. Rev. Lett.* **85** 968–71
[376] Pakonski P, Zyczkowski K and Kus M 2001 Classical 1d maps, quantum graphs and ensembles of unitary matrices *J. Phys. A: Math. Gen.* **34** 9303
[377] Berkolaiko G, Bogomolny E B and Keating J P 2001 Star graphs and Seba billiards *J. Phys. A: Math. Gen.* **34** 335–50
[378] Blümel R, Dabaghian Y and Jensen R 2002 Explicitly solvable cases of one-dimensional quantum chaos *Phys. Rev. Lett.* **88** 044101
[379] Pakoński P, Tanner G and Życzkowski K 2003 Families of line-graphs and their quantization *J. Stat. Phys.* **111** 1331–52
[380] Gnutzmann S and Altland A 2004 universal spectral statistics in quantum graphs *Phys. Rev. Lett.* **93** 194101

[381] Gnutzmann S and Smilansky U 2006 Quantum graphs: applications to quantum chaos and universal spectral statistics *Adv. Phys.* **55** 527–625
[382] Ławniczak M, Bauch S, Hul O and Sirko L 2010 Experimental investigation of the enhancement factor for microwave irregular networks with preserved and broken time reversal symmetry in the presence of absorption *Phys. Rev.* E **81** 046204
[383] Ławniczak M, Bauch S, Hul O and Sirko L 2011 Experimental investigation of the enhancement factor and the cross-correlation function for graphs with and without time-reversal symmetry: the open system case *Phys. Scr.* **2011** 014014
[384] Hul O, Ławniczak M, Bauch S, Sawicki A, Kuś M and Sirko L 2012 Are scattering properties of graphs uniquely connected to their shapes? *Phys. Rev. Lett.* **109** 040402
[385] Allgaier M, Gehler S, Barkhofen S, Stöckmann H-J and Kuhl U 2014 Spectral properties of microwave graphs with local absorption *Phys. Rev.* E **89** 022925
[386] Rehemanjiang A, Allgaier M, Joyner C H, Müller S, Sieber M, Kuhl U and Stöckmann H-J 2016 Microwave realization of the Gaussian symplectic ensemble *Phys. Rev. Lett.* **117** 064101
[387] Dietz B, Yunko V, Bialous M, Bauch S, Ławniczak M and Sirko L 2017 Nonuniversality in the spectral properties of time-reversal-invariant microwave networks and quantum graphs *Phys. Rev.* E **95** 052202
[388] Ławniczak M, Lipovský J C V and Sirko L 2019 Non-Weyl microwave graphs *Phys. Rev. Lett.* **122** 140503
[389] Białous M, Dietz B and Sirko L 2019 Experimental investigation of the elastic enhancement factor in a microwave cavity emulating a chaotic scattering system with varying openness *Phys. Rev.* E **100** 012210
[390] Simons B D and Altshuler B L 1993 universal velocity correlations in disordered and chaotic systems *Phys. Rev. Lett.* **70** 4063–6
[391] Aurich R, Bolte J and Steiner F 1994 universal signatures of quantum chaos *Phys. Rev. Lett.* **73** 1356–9
[392] Blum G, Gnutzmann S and Smilansky U 2002 Nodal domains statistics: a criterion for quantum chaos *Phys. Rev. Lett.* **88** 114101
[393] Relaño A, Gómez J M G, Molina R A, Retamosa J and Faleiro E 2002 Quantum chaos and $1/f$ noise *Phys. Rev. Lett.* **89** 244102
[394] Faleiro E, Gómez J M G, Molina R A, Muñoz L, Relaño A and Retamosa J 2004 Theoretical derivation of $1/f$ noise in quantum chaos *Phys. Rev. Lett.* **93** 244101
[395] Hemmady S, Zheng X, Ott E, Antonsen T M and Anlage S M 2005 universal impedance fluctuations in wave chaotic systems *Phys. Rev. Lett.* **94** 014102
[396] Pluhař Z and Weidenmüller H A 2014 universal quantum graphs *Phys. Rev. Lett.* **112** 144102
[397] Pal A and Huse D A 2010 Many-body localization phase transition *Phys. Rev.* B **82** 174411
[398] Jiang Y-Z, Chen Y-Y and Guan X-W 2015 Understanding many-body physics in one dimension from the Lieb-Liniger model *Chin. Phys.* B **24** 050311
[399] Nandkishore R and Huse D A 2015 Many-body localization and thermalization in quantum statistical mechanics *Annu. Rev. Condens. Matter Phys.* **6** 15–38
[400] Schreiber M, Hodgman S, Bordia P, Luschen H P, Fischer M H, Vosk R, Altman E, Schneider U and Bloch I 2015 Observation of many-body localization of interacting fermions in a quasirandom optical lattice *Science* **349** 842–5
[401] Choi J, Hild S, Zeiher J, Schaus P, Rubioabadal A, Yefsah T, Khemani V, Huse D A, Bloch I and Gross C 2016 Exploring the many-body localization transition in two dimensions *Science* **352** 1547–52
[402] Imbrie J Z 2016 On many-body localization for quantum spin chains *J. Stat. Phys.* **163** 998–1048
[403] Li H-B, Yang Y, Wang P and Wang X-G 2017 Identifying the closeness of eigenstates in quantum many-body systems *Chin. Phys.* B **26** 080502
[404] Alet F and Laflorencie N 2018 Many-body localization: an introduction and selected topics *C. R. Phys.* **19** 498–525
[405] Srednicki M 1994 Chaos and quantum thermalization *Phys. Rev.* E **50** 888
[406] Rigol M, Dunjko V and Olshanii M 2008 Thermalization and its mechanism for generic isolated quantum systems *Nature* **452** 854
[407] Rigol M and Santos L F 2010 Quantum chaos and thermalization in gapped systems *Phys. Rev.* A **82** 011604
[408] Alba V 2015 Eigenstate thermalization hypothesis and integrability in quantum spin chains *Phys. Rev.* B **91** 155123
[409] De Palma G, Serafini A, Giovannetti V and Cramer M 2015 Necessity of eigenstate thermalization *Phys. Rev. Lett.* **115** 220401
[410] Kaufman A M, Tai M E, Lukin A, Rispoli M, Schittko R, Preiss P M and Greiner M 2016 Quantum thermalization through entanglement in an isolated many-body system *Science* **353** 794–800
[411] D'Alessio L, Kafri Y, Polkovnikov A and Rigol M 2016 From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics *Adv. Phys.* **65** 239–362
[412] Tian C, Yang K, Fang P, Zhou H-J and Wang J 2018 Hidden thermal structure in fock space *Phys. Rev.* E **98** 060103
[413] Fang P, Zhao L and Tian C 2018 Concentration-of-measure theory for structures and fluctuations of waves *Phys. Rev. Lett.* **121** 140603
[414] Anza F, Gogolin C and Huber M 2018 Eigenstate thermalization for degenerate observables *Phys. Rev. Lett.* **120** 150603
[415] Deutsch J M 2018 Eigenstate thermalization hypothesis *Rep. Prog. Phys.* **81** 082001
[416] Xu J and Li Y 2019 Eigenstate distribution fluctuation of a quenched disordered bose-hubbard system in thermal-to-localized transitions *Chin. Phys. Lett.* **36** 027201
[417] Foini L and Kurchan J 2019 Eigenstate thermalization hypothesis and out of time order correlators *Phys. Rev.* E **99** 042139
[418] Lostaglio M, Korzekwa K, Jennings D and Rudolph T 2015 Quantum coherence, time-translation symmetry, and thermodynamics *Phys. Rev.* X **5** 021001
[419] Brandao F, Horodecki M, Ng N, Oppenheim J and Wehner S 2015 The second laws of quantum thermodynamics *Proc. Natl Acad. Sci.* **112** 3275–9
[420] Pekola J P 2015 Towards quantum thermodynamics in electronic circuits *Nat. Phys.* **11** 118
[421] Narasimhachar V and Gour G 2015 Low-temperature thermodynamics with quantum coherence *Nat. Commun.* **6** 7689
[422] Binder F, Vinjanampathy S, Modi K and Goold J 2015 Quantum thermodynamics of general quantum processes *Phys. Rev.* E **91** 032119
[423] Millen J and Xuereb A 2016 Perspective on quantum thermodynamics *New J. Phys.* **18** 011002
[424] Vinjanampathy S and Anders J 2016 Quantum thermodynamics *Contemp. Phys.* **57** 545–79
[425] Campisi M and Goold J 2017 Thermodynamics of quantum information scrambling *Phys. Rev.* E **95** 062127
[426] Binder F, Correa L A, Gogolin C, Anders J and Adesso G 2018 Thermodynamics in the quantum regime *Fundamental Theories of Physics* (Berlin: Springer)
[427] Aleiner I L, Faoro L and Ioffe L B 2016 Microscopic model of quantum butterfly effect: out-of-time-order correlators and traveling combustion waves *Ann. Phys.* **375** 378–406

[428] Rozenbaum E B, Galitski V and Ganeshan S 2017 Lyapunov exponent and out-of-time-ordered correlator's growth rate in a chaotic system *Phys. Rev. Lett.* **118** 086801
[429] Halpern N Y 2017 Jarzynski-like equality for the out-of-time-ordered correlator *Phys. Rev.* A **95** 012120
[430] Huang Y, Zhang Y-L and Chen X 2017 Out-of-time-ordered correlators in many-body localized systems *Ann. Phys.* **529** 1600318
[431] Li J, Fan R, Wang H, Ye B, Zeng B, Zhai H, Peng X and Du J 2017 Measuring out-of-time-order correlators on a nuclear magnetic resonance quantum simulator *Phys. Rev.* X **7** 031011
[432] Halpern N Y, Swingle B and Dressel J 2018 Quasiprobability behind the out-of-time-ordered correlator *Phys. Rev.* A **97** 042105
[433] Garciamata I, Jalabert R A, Saraceno M, Roncaglia A J and Wisniacki D A 2018 Chaos signatures in the short and long time behavior of the out-of-time ordered correlator *Phys. Rev. Lett.* **121** 210601
[434] Rakovszky T, Pollmann F and Von Keyserlingk C 2018 Diffusive hydrodynamics of out-of-time-ordered correlators with charge conservation *Phys. Rev.* X **8** 031058
[435] Alonso J R G, Halpern N Y and Dressel J 2019 Out-of-time-ordered-correlator quasiprobabilities robustly witness scrambling *Phys. Rev. Lett.* **122** 040404
[436] Lakshminarayan A 2019 Out-of-time-ordered correlator in the quantum bakers map and truncated unitary matrices *Phys. Rev.* E **99** 012201
[437] Chen Y and Tian C 2014 Planck's quantum-driven integer quantum Hall effect in chaos *Phys. Rev. Lett.* **113** 216802
[438] Tian C, Chen Y and Wang J 2016 Emergence of integer quantum Hall effect from chaos *Phys. Rev.* B **93** 075403
[439] Zhao W-L, Jie Q-L and Zhou B 2010 Quantum to classical transition by a classically small interaction *Commun. Theor. Phys.* **54** 247
[440] Xu Y-Y 2013 Interference of quantum chaotic systems in phase space *Commun. Theor. Phys.* **60** 453
[441] Zhang Y-H, Zhang J-Q, Xu X-Y and Lin S-L 2009 The quantum spectral analysis of the two-dimensional annular billiard system *Chin. Phys.* B **18** 35–9
[442] Yu H, Ren Z and Zhang X 2019 Dynamical stable-jump-stable-jump picture in a non-periodically driven quantum relativistic kicked rotor system *Chin. Phys.* B **28** 20504
[443] Liu C-R, Yu P, Chen X-Z, Xu H-Y, Huang L and Lai Y-C 2019 Enhancing von neumann entropy by chaos in spin-orbit entanglement *Chin. Phys.* B **28** 100501
[444] Xu X-Y, Gao S, Guo W-H, Zhang Y-H and Lin S-L 2006 Semiclassical analysis of quarter stadium billiards *Chin. Phys. Lett.* **23** 765–7
[445] Tan J-T, Luo Y-R, Zhou Z and Hai W-H 2016 Combined effect of classical chaos and quantum resonance on entanglement dynamics *Chin. Phys. Lett.* **33** 070302
[446] Ge M, Zhang Y, Wang D, Du M and Lin S 2005 The dynamical properties of Rydberg hydrogen atom near a metal surface *Sci. China-Phys. Mech. Astron.* **48** 667–75
[447] Li H, Gao S, Xu X and Lin S 2008 Scattering matrix theory for Cs Rydberg atoms in magnetic field *Sci. China-Phys. Mech. Astron.* **51** 499–506
[448] Xin J and Liang J 2014 Exact solutions of a spin–orbit coupling model in two-dimensional central-potentials and quantum–classical correspondence *Sci. China-Phys. Mech. Astron.* **57** 1504–10
[449] Zhao Y and Wu B 2019 Quantum-classical correspondence in integrable systems *Sci. China-Phys. Mech. Astron.* **62** 997011
[450] Berry M V and Mondragon R J 1987 Neutrino billiards—time-reversal symmetry-breaking without magnetic-fields *Proc. R. Soc.* A **412** 53–74
[451] Chodos A, Jaffe R L, Johnson K and Thorn C B 1974 Baryon structure in the bag theory *Phys. Rev.* D **10** 2599–604
[452] Antoine M, Comtet A and Knecht M 1990 Heat kernel expansion for fermionic billiards in an external magnetic field *J. Phys. A: Math. Gen.* **23** L35
[453] Phatak S C, Pal S and Biswas D 1995 Semiclassical features in the quantum description of a Dirac particle in a cavity *Phys. Rev.* E **52** 1333–44
[454] Novoselov K S, Geim A K, Morozov S V, Jiang D, Zhang Y, Dubonos S V, Grigorieva I V and Firsov A A 2004 Electric field effect in atomically thin carbon films *Science* **306** 666–9
[455] Novoselov K S, Geim A K, Morozov S V, Jiang D, Katsnelson M I, Grigorieva I V, Dubonos S V and Firsov A A 2005 Two-dimensional gas of massless Dirac fermions in graphene *Nature* **438** 197–200
[456] Beenakker C W J 2008 Colloquium: Andreev reflection and Klein tunneling in graphene *Rev. Mod. Phys.* **80** 1337–54
[457] Castro Neto A H, Guinea F, Peres N M R, Novoselov K S and Geim A K 2009 The electronic properties of graphene *Rev. Mod. Phys.* **81** 109–62
[458] Hasan M Z and Kane C L 2010 Colloquium: topological insulators *Rev. Mod. Phys.* **82** 3045–67
[459] Qi X-L and Zhang S-C 2011 Topological insulators and superconductors *Rev. Mod. Phys.* **83** 1057–110
[460] Wehling T, Black-Schaffer A and Balatsky A 2014 Dirac materials *Adv. Phys.* **63** 1–76
[461] Miao F, Wijeratne S, Zhang Y, Coskun U, Bao W and Lau C 2007 Phase-coherent transport in graphene quantum billiards *Science* **317** 1530–3
[462] Ponomarenko L, Schedin F, Katsnelson M, Yang R, Hill E, Novoselov K and Geim A 2008 Chaotic Dirac billiard in graphene quantum dots *Science* **320** 356–8
[463] Lai Y-C, Xu H-Y, Huang L and Grebogi C 2018 Relativistic quantum chaos: an emergent interdisciplinary field *Chaos* **28** 052101
[464] Huang L, Xu H-Y, Grebogi C and Lai Y-C 2018 Relativistic quantum chaos *Phys. Rep.* **753** 1–128
[465] Barrow J D 1982 General relativistic chaos and nonlinear dynamics *Gen. Relat. Grav.* **14** 523–30
[466] Barrow J D 1982 Chaotic behaviour in general relativity *Phys. Rep.* **85** 1–49
[467] Lockhart C M, Misra B and Prigogine I 1982 Geodesic instability and internal time in relativistic cosmology *Phys. Rev.* D **25** 921–9
[468] Chernikov A A, Tél T, Vattay G and Zaslavsky G M 1989 Chaos in the relativistic generalization of the standard map *Phys. Rev.* A **40** 4072–6
[469] Nomura Y, Ichikawa Y H and Horton W 1992 Nonlinear dynamics of the relativistic standard map *Phys. Rev.* A **45** 1103–15
[470] Kim J-H and Lee H-W 1995 Relativistic chaos in the driven harmonic oscillator *Phys. Rev.* E **51** 1579–81
[471] Drake S P, Dettmann C P, Frankel N E and Cornish N J 1996 Chaos in special relativistic dynamics *Phys. Rev.* E **53** 1351–61
[472] Podolský J V and Veselý K 1998 Chaos in pp-wave spacetimes *Phys. Rev.* D **58** 081501
[473] Barrow J D and Levin J 1998 Chaos in the Einstein-Yang-Mills equations *Phys. Rev. Lett.* **80** 656–9
[474] Tomaschitz R 2000 Tachyons, Lamb shifts and superluminal chaos *Euro. Phys. J.* B **17** 523–36
[475] Burnell F, Mann R B and Ohta T 2003 Chaos in a relativistic 3-body self-gravitating system *Phys. Rev. Lett.* **90** 134101
[476] Motter A E 2003 Relativistic chaos is coordinate invariant *Phys. Rev. Lett.* **91** 231101
[477] Motter A E and Saa A 2009 Relativistic invariance of Lyapunov exponents in bounded and unbounded systems *Phys. Rev. Lett.* **102** 184101

[478] Bernal J D, Seoane J M and Sanjuán M A F 2017 Global relativistic effects in chaotic scattering *Phys. Rev.* E **95** 032205
[479] Bernal J D, Seoane J M and Sanjuán M A F 2018 Uncertainty dimension and basin entropy in relativistic chaotic scattering *Phys. Rev.* E **97** 042214
[480] Hou X-W and Hu B 2004 Decoherence, entanglement, and chaos in the Dicke model *Phys. Rev.* A **69** 042110
[481] Santos L F, Rigolin G and Escobar C O 2004 Entanglement versus chaos in disordered spin chains *Phys. Rev.* A **69** 042304
[482] Mejía-Monasterio C, Benenti G, Carlo G G and Casati G 2005 Entanglement across a transition to quantum chaos *Phys. Rev.* A **71** 062324
[483] Wang G-L, Huang L, Lai Y-C and Grebogi C 2014 Nonlinear dynamics and quantum entanglement in optomechanical systems *Phys. Rev. Lett.* **112** 110406
[484] Pershin Y V and Privman V 2004 Low spin relaxation in two-dimensional electron systems with antidots *Phys. Rev.* B **69** 073310
[485] Zaitsev O, Frustaglia D and Richter K 2004 Role of orbital dynamics in spin relaxation and weak antilocalization in quantum dots *Phys. Rev. Lett.* **94** 026809
[486] Chang C H, Mal'shukov A G and Chao K A 2004 Spin relaxation dynamics of quasiclassical electrons in ballistic quantum dots with strong spin–orbit coupling *Phys. Rev.* B **70** 245309
[487] Akguc G B and Gong J 2008 Spin-dependent electron transport in two-dimensional waveguides of arbitrary geometry *Phys. Rev.* B **77** 205302
[488] Ying L and Lai Y-C 2016 Enhancement of spin polarization by chaos in graphene quantum dot systems *Phys. Rev.* B **93** 085408
[489] Liu C-R, Chen X-Z, Xu H-Y, Huang L and Lai Y-C 2018 Effect of chaos on two-dimensional spin transport *Phys. Rev.* B **98** 115305
[490] Tomaschitz R 1991 Relativistic quantum chaos in Robertson-Walker cosmologies *J. Math. Phys.* **32** 2571–9
[491] Berger B K 1989 Quantum chaos in the mixmaster universe *Phys. Rev.* D **39** 2426–9
[492] Kirillov A A and Melnikov V N 1995 Dynamics of inhomogeneities of the metric in the vicinity of a singularity in multidimensional cosmology *Phys. Rev.* D **52** 723–9
[493] Calzetta E and Gonzalez J J 1995 Chaos and semiclassical limit in quantum cosmology *Phys. Rev.* D **51** 6821–8
[494] Cornish N J and Shellard E P S 1998 Chaos in quantum cosmology *Phys. Rev. Lett.* **81** 3571–4
[495] Damour T and Henneaux M 2001 E_{10}, BE_{10} and arithmetical chaos in superstring cosmology *Phys. Rev. Lett.* **86** 4749–52
[496] Bojowald M and Date G 2004 Quantum suppression of the generic chaotic behavior close to cosmological singularities *Phys. Rev. Lett.* **92** 071302
[497] Kleinschmidt A, Koehn M and Nicolai H 2009 Supersymmetric quantum cosmological billiards *Phys. Rev.* D **80** 061701
[498] Koehn M 2012 Relativistic wavepackets in classically chaotic quantum cosmological billiards *Phys. Rev.* D **85** 063501
[499] Leutwyler H and Smilga A 1992 Spectrum of Dirac operator and role of winding number in QCD *Phys. Rev.* D **46** 5607–32
[500] Shuryak E and Verbaarschot J J M 1993 Random matrix theory and spectral sum rules for the Dirac operator in QCD *Nucl. Phys.* **560** 306–20
[501] Verbaarschot J J M and Zahed I 1993 Spectral density of the QCD Dirac operator near zero virtuality *Phys. Rev. Lett.* **70** 3852–5
[502] Verbaarschot J 1994 Spectrum of the QCD Dirac operator and chiral random matrix theory *Phys. Rev. Lett.* **72** 2531–3
[503] Halasz M A and Verbaarschot J J M 1995 universal fluctuations in spectra of the lattice Dirac operator *Phys. Rev. Lett.* **74** 3920–3
[504] Berg B A, Markum H and Pullirsch R 1999 Quantum chaos in compact lattice QED *Phys. Rev.* D **59** 097504
[505] Akemann G and Kanzieper E 2000 Spectra of massive and massless QCD Dirac operators: a novel link *Phys. Rev. Lett.* **85** 1174–7
[506] Toublan D and Verbaarschot J 2001 Statistical properties of the spectrum of the QCD Dirac operator at low energy *Nucl. Phys.* B **603** 343– 368
[507] Beenakker C W J 2015 Random-matrix theory of Majorana fermions and topological superconductors *Rev. Mod. Phys.* **87** 1037–66
[508] Bolte J and Harrison J 2003 Spectral statistics for the Dirac operator on graphs *J. Phys. A: Math. Gen.* **36** 2747–69
[509] Harrison J M, Weyand T and Kirsten K 2016 Zeta functions of the Dirac operator on quantum graphs *J. Math. Phys.* **57** 102301
[510] Matrasulov D U, Milibaeva G M, Salomov U R and Sundaram B 2005 Relativistic kicked rotor *Phys. Rev.* E **72** 016213
[511] Zhao Q, Müller C A and Gong J 2014 Quantum and classical superballistic transport in a relativistic kicked-rotor system *Phys. Rev.* E **90** 022921
[512] Rozenbaum E B and Galitski V 2017 Dynamical localization of coupled relativistic kicked rotors *Phys. Rev.* B **95** 064303
[513] Pauli W 1932 Diracs wellengleichung des elektrons und geometrische optik *Helv. Phys. Acta* **5** 447
[514] Rubinow S I and Keller J B 1963 Asymptotic solution of the Dirac equation *Phys. Rev.* **131** 2789–96
[515] Yabana K and Horiuchi H 1986 Adiabatic viewpoint for the WKB treatment of coupled channel system appearance of the Berry phase and another extra phase accompanying the adiabatic motion *Prog. Theor. Phys.* **75** 592–618
[516] Berry M V and Wilkinson M 1984 Diabolical points in the spectra of triangles *Proc. R. Soc.* A **392** 15–43
[517] Wilczek F and Shapere A 1989 *Geometric Phases in Physics* vol 5 (Singapore: World Scientific)
[518] Kuratsuji H and Iida S 1985 Effective action for adiabatic process dynamical meaning of Berry and Simon's phase *Prog. Theor. Phys.* **74** 439–45
[519] Kuratsuji H and Iida S 1988 Deformation of symplectic structure and anomalous commutators in field theories *Phys. Rev.* D **37** 441–7
[520] Littlejohn R G and Flynn W G 1991 Geometric phases in the asymptotic theory of coupled wave equations *Phys. Rev.* A **44** 5239–56
[521] Littlejohn R G and Flynn W G 1991 Geometric phases and the Bohr–Sommerfeld quantization of multicomponent wave fields *Phys. Rev. Lett.* **66** 2839–42
[522] Emmrich C and Weinstein A 1996 Geometry of the transport equation in multicomponent WKB approximations *Commun. Math. Phys.* **176** 701–11
[523] Yajima K 1982 The quasiclassical approximation to Dirac equation. I *J. Fac Sci. Univ. Tokyo Math.* **29** 161–94 Sect. 1 A
[524] Bagrov V G, Belov V V, Trifonov A Y and Yevseyevich A A 1994 Quantization of closed orbits in Dirac theory by Maslov's complex germ method *J. Phys. A: Math. Gen.* **27** 1021–43
[525] Bagrov V G, Belov V V, Trifonov A Y and Yevseyevicht A A 1994 Quasi-classical spectral series of the Dirac operators corresponding to quantized two-dimensional Lagrangian tori *J. Phys. A: Math. Gen.* **27** 5273–306
[526] Spohn H 2000 Semiclassical limit of the Dirac equation and spin precession *Ann. Phys., NY* **282** 420–31
[527] Bolte J and Keppeler S 1999 A semiclassical approach to the Dirac equation *Ann. Phys., NY* **274** 125–62

[528] Bolte J and Keppeler S 1999 Semiclassical form factor for chaotic systems with spin $1/2$ *J. Phys. A: Math. Gen.* **32** 8863
[529] Bolte J and Keppeler S 1998 Semiclassical time evolution and trace formula for relativistic spin-$1/2$ particles *Phys. Rev. Lett.* **81** 1987–91
[530] Bolte J, Glaser R and Keppeler S 2001 Quantum and classical ergodicity of spinning particles *Ann. Phys., NY* **293** 1–14
[531] Keppeler S 2003 Semiclassical quantisation rules for the Dirac and Pauli equations *Ann. Phys., NY* **304** 40–71
[532] Wurm J, Richter K and Adagideli I 2011 Edge effects in graphene nanostructures: from multiple reflection expansion to density of states *Phys. Rev.* B **84** 075468
[533] Wurm J, Richter K and Adagideli I 2011 Edge effects in graphene nanostructures: semiclassical theory of spectral fluctuations and quantum transport *Phys. Rev.* B **84** 205421
[534] Huang L, Lai Y-C and Grebogi C 2011 Characteristics of level-spacing statistics in chaotic graphene billiards *Chaos* **21** 013102
[535] Huang L, Xu H-Y, Lai Y-C and Grebogi C 2014 Level spacing statistics for two-dimensional massless Dirac billiards *Chin. Phys.* B **23** 070507
[536] Ni X, Huang L, Lai Y-C and Grebogi C 2012 Scarring of Dirac fermions in chaotic billiards *Phys. Rev.* E **86** 016702
[537] Yu P, Dietz B and Huang L 2019 Quantizing neutrino billiards: an expanded boundary integral method *New J. Phys.* **21** 073039
[538] Wurm J, Rycerz A, Adagideli İ, Wimmer M, Richter K and Baranger H U 2009 Symmetry classes in graphene quantum dots: universal spectral statistics, weak localization, and conductance fluctuations *Phys. Rev. Lett.* **102** 056806
[539] Huang L, Lai Y-C and Grebogi C 2010 Relativistic quantum level-spacing statistics in chaotic graphene billiards *Phys. Rev.* E **81** 055203
[540] Dietz B, Klaus T, Miski-Oglu M, Richter A, Wunderle M and Bouazza C 2016 Spectral properties of Dirac billiards at the van Hove singularities *Phys. Rev. Lett.* **116** 023901
[541] Tan C-L, Tan Z-B, Ma L, Chen J, Yang F, Qu F-M, Liu G-T, Yang H-F, Yang C-L and Lü L 2009 Quantum chaos in graphene nanoribbon quantum dot *Acta Phys. Sin.* **58** 5726
[542] Libisch F, Stampfer C and Burgdörfer J 2009 Graphene quantum dots: beyond a Dirac billiard *Phys. Rev.* B **79** 115423
[543] Amanatidis I and Evangelou S N 2009 Quantum chaos in weakly disordered graphene *Phys. Rev.* B **79** 205420
[544] Rycerz A 2012 Random matrices and quantum chaos in weakly disordered graphene nanoflakes *Phys. Rev.* B **85** 245424
[545] Wimmer M, Akhmerov A R and Guinea F 2010 Robustness of edge states in graphene quantum dots *Phys. Rev.* B **82** 045409
[546] Rycerz A 2013 Strain-induced transitions to quantum chaos and effective time-reversal symmetry breaking in triangular graphene nanoflakes *Phys. Rev.* B **87** 195431
[547] Zhang D-B, Seifert G and Chang K 2014 Strain-induced pseudomagnetic fields in twisted graphene nanoribbons *Phys. Rev. Lett.* **112** 096805
[548] Liu Z, Zhang D-B, Seifert G, Liu Y and Chang K 2019 Interfacial Landau levels in bent graphene racetracks *Phys. Rev.* B **99** 165416
[549] Polini M, Guinea F, Lewenstein M, Manoharan H C and Pellegrini V 2013 Artificial honeycomb lattices for electrons, atoms and photons *Nat. Nanotechnol.* **8** 625–33
[550] Zandbergen S R and de Dood M J A 2010 Experimental observation of strong edge effects on the pseudodiffusive transport of light in photonic graphene *Phys. Rev. Lett.* **104** 043903
[551] Bittner S, Dietz B, Miski-Oglu M, Oria Iriarte P, Richter A and Schäfer F 2010 Observation of a Dirac point in microwave experiments with a photonic crystal modeling graphene *Phys. Rev.* B **82** 014301
[552] Kuhl U, Barkhofen S, Tudorovskiy T, Stöckmann H-J, Hossain T, de Forges de Parny L and Mortessagne F 2010 Dirac point and edge states in a microwave realization of tight-binding graphene-like structures *Phys. Rev.* B **82** 094308
[553] Bellec M, Kuhl U, Montambaux G and Mortessagne F 2013 Topological transition of Dirac points in a microwave experiment *Phys. Rev. Lett.* **110** 033902
[554] Poo Y, Wu R-X, Lin Z, Yang Y and Chan C T 2011 Experimental realization of self-guiding unidirectional electromagnetic edge states *Phys. Rev. Lett.* **106** 093903
[555] Bittner S, Dietz B, Miski-Oglu M and Richter A 2012 Extremal transmission through a microwave photonic crystal and the observation of edge states in a rectangular Dirac billiard *Phys. Rev.* B **85** 064301
[556] Bellec M, Kuhl U, Montambaux G and Mortessagne F 2013 Tight-binding couplings in microwave artificial graphene *Phys. Rev.* B **88** 115437
[557] Wang X, Jiang H T, Yan C, Sun Y, Li Y H, Shi Y L and Chen H 2013 Anomalous transmission of disordered photonic graphenes at the Dirac point *Europhys. Lett.* **103** 17003
[558] Plotnik Y *et al* 2013 Observation of unconventional edge states in photonic graphene *Nat. Mater.* **13** 57–62
[559] Dietz B, Iachello F, Miski-Oglu M, Pietralla N, Richter A, von Smekal L and Wambach J 2013 Lifshitz and excited-state quantum phase transitions in microwave Dirac billiards *Phys. Rev.* B **88** 104101
[560] Wang X, Jiang H T, Yan C, Deng F S, Sun Y, Li Y H, Shi Y L and Chen H 2014 Transmission properties near Dirac-like point in two-dimensional dielectric photonic crystals *Europhys. Lett.* **108** 14002
[561] Wang X, Jiang H, Li Y, Yan C, Deng F, Sun Y, Li Y, Shi Y and Chen H 2015 Transport properties of disordered photonic crystals around a Dirac-like point *Opt. Exp.* **23** 5126–33
[562] Dietz B, Klaus T, Miski-Oglu M and Richter A 2015 Spectral properties of superconducting microwave photonic crystals modeling Dirac billiards *Phys. Rev.* B **91** 035411
[563] Dietz B and Richter A 2019 From graphene to fullerene: experiments with microwave photonic crystals *Phys. Scr.* **94** 014002
[564] Yu P, Li Z-Y, Xu H-Y, Huang L, Dietz B, Grebogi C and Lai Y-C 2016 Gaussian orthogonal ensemble statistics in graphene billiards with the shape of classically integrable billiards *Phys. Rev.* E **94** 062214
[565] Huang L, Lai Y-C, Ferry D K, Goodnick S M and Akis R 2009 Relativistic quantum scars *Phys. Rev. Lett.* **103** 054101
[566] Mason D J, Borunda M F and Heller E J 2013 Semiclassical deconstruction of quantum states in graphene *Phys. Rev.* B **88** 165421
[567] Cabosart D, Felten A, Reckinger N, Iordanescu A, Toussaint S, Faniel S and Hackens B 2017 Recurrent quantum scars in a mesoscopic graphene ring *Nano Lett.* **17** 1344–9
[568] Xu H Y, Huang L, Lai Y-C and Grebogi C 2013 Chiral scars in chaotic Dirac fermion systems *Phys. Rev. Lett.* **110** 064102
[569] Wang C-Z, Huang L and Chang K 2017 Scars in Dirac fermion systems: the influence of an Aharonov–Bohm flux *New J. Phys.* **19** 013018
[570] Song M-Y, Li Z-Y, Xu H-Y, Huang L and Lai Y-C 2019 Quantization of massive Dirac billiards and unification of nonrelativistic and relativistic chiral quantum scars *Phys. Rev. Res.* **1** 033008
[571] Bardarson J H, Titov M and Brouwer P W 2009 Electrostatic confinement of electrons in an integrable graphene quantum dot *Phys. Rev. Lett.* **102** 226803

[572] Schneider M and Brouwer P W 2011 Resonant scattering in graphene with a gate-defined chaotic quantum dot *Phys. Rev.* B **84** 115440
[573] Yang R, Huang L, Lai Y-C and Grebogi C 2011 Quantum chaotic scattering in graphene systems *Europhys. Lett.* **94** 40004
[574] Barros M S M, Júnior A J N, Macedo-Junior A F, Ramos J G G S and Barbosa A L R 2013 Open chaotic Dirac billiards: weak (anti)localization, conductance fluctuations, and decoherence *Phys. Rev.* B **88** 245133
[575] Schomerus H, Marciani M and Beenakker C W J 2015 Effect of chiral symmetry on chaotic scattering from Majorana zero modes *Phys. Rev. Lett.* **114** 166803
[576] Ramos J G G S, Hussein M S and Barbosa A L R 2016 Fluctuation phenomena in chaotic Dirac quantum dots: Artificial atoms on graphene flakes *Phys. Rev.* B **93** 125136
[577] Vasconcelos T C, Ramos J G G S and Barbosa A L R 2016 universal spin Hall conductance fluctuations in chaotic Dirac quantum dots *Phys. Rev.* B **93** 115120
[578] Datta S 1995 *Electronic Transport in Mesoscopic systems* (Cambridge: Cambridge University Press)
[579] Li T C and Lu S-P 2008 Quantum conductance of graphene nanoribbons with edge defects *Phys. Rev.* B **77** 085408
[580] Huang L, Lai Y, Ferry D K, Akis R and Goodnick S M 2009 Transmission and scarring in graphene quantum dots *J. Phys.: Condens. Matter* **21** 344203
[581] Zhang S-H, Yang W and Chang K 2017 General Green's function formalism for layered systems: Wave function approach *Phys. Rev.* B **95** 075421
[582] Huang L, Lai Y-C, Luo H-G and Grebogi C 2015 universal formalism of Fano resonance *AIP Adv.* **5** 017137
[583] Yang R, Huang L, Lai Y-C and Pecora L M 2012 Modulating quantum transport by transient chaos *Appl. Phys. Lett.* **100** 093105
[584] Yang R, Huang L, Lai Y-C, Grebogi C and Pecora L M 2013 Harnessing quantum transport by transient chaos *Chaos* **23** 013125
[585] Ying L, Huang L, Lai Y-C and Grebogi C 2012 Conductance fluctuations in graphene systems: the relevance of classical dynamics *Phys. Rev.* B **85** 245448
[586] Bao R, Huang L, Lai Y-C and Grebogi C 2015 Conductance fluctuations in chaotic bilayer graphene quantum dots *Phys. Rev.* E **92** 012918
[587] Büttiker M, Imry Y and Landauer R 1983 Josephson behavior in small normal one-dimensional rings *Phys. Lett.* A **96** 365–7
[588] Sheng J S and Chang K 2006 Spin states and persistent currents in mesoscopic rings: spin–orbit interactions *Phys. Rev.* B **74** 235315
[589] Aharonov Y and Bohm D 1959 Significance of electromagnetic potentials in the quantum theory *Phys. Rev.* **115** 485
[590] Lévy L P, Dolan G, Dunsmuir J and Bouchiat H 1990 Magnetization of mesoscopic copper rings: evidence for persistent currents *Phys. Rev. Lett.* **64** 2074–7
[591] Chandrasekhar V, Webb R A, Brady M J, Ketchen M B, Gallagher W J and Kleinsasser A 1991 Magnetic response of a single, isolated gold loop *Phys. Rev. Lett.* **67** 3578–81
[592] Bleszynski-Jayich1 A C, Shanks W E, Peaudecerf B, Ginossar E, von Oppen F, Glazman L and Harris J G E 2009 Persistent currents in normal metal rings *Science* **326** 272–5
[593] Bluhm H, Koshnick N C, Bert J A, Huber M E and Moler K A 2009 Persistent currents in normal metal rings *Phys. Rev. Lett.* **102** 136802
[594] Mailly D, chelier C and Benoit A 1993 Experimental observation of persistent currents in GaAs-AlGaAs single loop *Phys. Rev. Lett.* **70** 2020–3
[595] Chakraborty T and Pietiläinen P 1995 Persistent currents in a quantum ring: effects of impurities and interactions *Phys. Rev.* B **52** 1932–5
[596] Kawabata S 1999 Persistent currents in quantum chaotic systems *Phys. Rev.* B **59** 12256–9
[597] Pershin Y V and Piermarocchi C 2005 Persistent and radiation-induced currents in distorted quantum rings *Phys. Rev.* B **72** 125348
[598] Bruno-Alfonso A and Latgé A 2008 Quantum rings of arbitrary shape and non-uniform width in a threading magnetic field *Phys. Rev.* B **77** 205303
[599] Chang K and Lou W-K 2011 Helical quantum states in HgTe quantum dots with inverted band structures *Phys. Rev. Lett.* **106** 206802
[600] Li J, Lou W-K, Zhang D, Li X-J, Yang W and Chang K 2014 Single- and few-electron states in topological-insulator quantum dots *Phys. Rev.* B **90** 115303
[601] Zhao Y, Wyrick J, Natterer F D, Rodriguez-Nieva J F, Lewandowski C, Watanabe K, Taniguchi T, Levitov L S, Zhitenev N B and Stroscio J A 2015 Creating and probing electron whispering-gallery modes in graphene *Science* **348** 672–5
[602] Xu H-Y, Huang L, Lai Y-C and Grebogi C 2015 Superpersistent currents and whispering gallery modes in relativistic quantum chaotic systems *Sci. Rep.* **5** 8963
[603] Xu H-Y, Huang L and Lai Y-C 2016 A robust relativistic quantum two-level system with edge-dependent currents and spin polarization *Europhys. Lett.* **115** 20005
[604] Ying L and Lai Y-C 2017 Robustness of persistent currents in two-dimensional Dirac systems with disorder *Phys. Rev.* B **96** 165407
[605] Han C-D, Xu H-Y, Huang L and Lai Y-C 2019 Manifestations of chaos in relativistic quantum systems—a study based on out-of-time-order correlator *Phys. Open* **1** 100001
[606] Sutherland B 1986 Localization of electronic wave functions due to local topology *Phys. Rev.* B **34** 5208–11
[607] Bercioux D, Urban D F, Grabert H and Häusler W 2009 Massless Dirac-Weyl fermions in a T_3 optical lattice *Phys. Rev.* A **80** 063603
[608] Shen R, Shao L B, Wang B and Xing D Y 2010 Single Dirac cone with a flat band touching on line-centered-square optical lattices *Phys. Rev.* B **81** 041410
[609] Green D, Santos L and Chamon C 2010 Isolated flat bands and spin-1 conical bands in two-dimensional lattices *Phys. Rev.* B **82** 075104
[610] Dóra B, Kailasvuori J and Moessner R 2011 Lattice generalization of the Dirac equation to general spin and the role of the flat band *Phys. Rev.* B **84** 195422
[611] Wang F and Ran Y 2011 Nearly flat band with Chern number $c = 2$ on the dice lattice *Phys. Rev.* B **84** 241103
[612] Huang X, Lai Y, Hang Z H, Zheng H and Chan C T 2011 Dirac cones induced by accidental degeneracy in photonic crystals and zero-refractive-index materials *Nat. Mater.* **10** 582–6
[613] Mei J, Wu Y, Chan C T and Zhang Z-Q 2012 First-principles study of Dirac and Dirac-like cones in phononic and photonic crystals *Phys. Rev.* B **86** 035141
[614] Moitra P, Yang Y-M, Anderson Z, Kravchenko I I, Briggs D P and Valentine J 2013 Realization of an all-dielectric zero-index optical metamaterial *Nat. Photon.* **7** 791–5
[615] Raoux A, Morigi M, Fuchs J-N, Piéchon F and Montambaux G 2014 From dia- to paramagnetic orbital susceptibility of massless fermions *Phys. Rev. Lett.* **112** 026402
[616] Guzmán-Silva D, Mejía-Cortés C, Bandres M A, Rechtsman M C, Weimann S, Nolte S, Segev M, Szameit A and Vicencio R A 2014 Experimental observation of bulk and edge transport in photonic Lieb lattices *New J. Phys.* **16** 063061

[617] Romhányi J, Penc K and Ganesh R 2015 Hall effect of triplons in a dimerized quantum magnet *Nat. Commun.* **6** 6805

[618] Giovannetti G, Capone M, van den Brink J and Ortix C 2015 Kekulé textures, pseudospin-one Dirac cones, and quadratic band crossings in a graphene-hexagonal indium chalcogenide bilayer *Phys. Rev.* B **91** 121417

[619] Li Y, Kita s, Munoz P, Reshef O, Vulis D I, Yin M, Loncar M and Mazur E 2015 On-chip zero-index metamaterials *Nat. Photon.* **9** 738–42

[620] Mukherjee S, Spracklen A, Choudhury D, Goldman N, Öhberg P, Andersson E and Thomson R R 2015 Observation of a localized flat-band state in a photonic Lieb lattice *Phys. Rev. Lett.* **114** 245504

[621] Vicencio R A, Cantillano C, Morales-Inostroza L, Real B, Mejía-Cortés C, Weimann S, Szameit A and Molina M I 2015 Observation of localized states in Lieb photonic lattices *Phys. Rev. Lett.* **114** 245503

[622] Taie S, Ozawa H, Ichinose T, Nishio T, Nakajima S and Takahashi Y 2015 Coherent driving and freezing of bosonic matter wave in an optical Lieb lattice *Sci. Adv.* **1** e1500854

[623] Fang A, Zhang Z Q, Louie S G and Chan C T 2016 Klein tunneling and supercollimation of pseudospin-1 electromagnetic waves *Phys. Rev.* B **93** 035422

[624] Diebel F, Leykam D, Kroesen S, Denz C and Desyatnikov A S 2016 Conical diffraction and composite Lieb Bosons in photonic lattices *Phys. Rev. Lett.* **116** 183902

[625] Zhu L, Wang S-S, Guan S, Liu Y, Zhang T, Chen G and Yang S A 2016 Blue phosphorene oxide: strain-tunable quantum phase transitions and novel 2D emergent fermions *Nano Lett.* **16** 6548–54

[626] Bradlyn B, Cano J, Wang Z, Vergniory M G, Felser C, Cava R J and Bernevig B A Beyond Dirac and Weyl fermions: unconventional quasiparticles in conventional crystals *Science* **353** 1–7

[627] Fulga I C and Stern A 2017 Triple point fermions in a minimal symmorphic model *Phys. Rev.* B **95** 241116

[628] Ezawa M 2017 Triplet fermions and Dirac fermions in borophene *Phys. Rev.* B **96** 035425

[629] Zhong C, Chen Y, Yu Z-M, Xie Y, Wang H, Yang S A and Zhang S 2017 Three-dimensional pentagon carbon with a genesis of emergent fermions *Nat. Commun.* **8** 15641

[630] Zhu Y-Q, Zhang D-W, Yan H, Xing D-Y and Zhu S-L 2017 Emergent pseudospin-1 Maxwell fermions with a threefold degeneracy in optical lattices *Phys. Rev.* A **96** 033634

[631] Drost R, Ojanen T, Harju A and Liljeroth P 2017 Topological states in engineered atomic lattices *Nat. Phys.* **13** 668

[632] Slot M R, Gardenier T S, Jacobse P H, van Miert G C P, Kempkes S N, Zevenhuizen S J M, Smith C M, Vanmaekelbergh D and Swart I 2017 Experimental realization and characterization of an electronic Lieb lattice *Nat. Phys.* **13** 672

[633] Tan X, Zhang D-W, Liu Q, Xue G, Yu H-F, Zhu Y-Q, Yan H, Zhu S-L and Yu Y 2018 Topological Maxwell metal bands in a superconducting qutrit *Phys. Rev. Lett.* **120** 130503

[634] Urban D F, Bercioux D, Wimmer M and Häusler W 2011 Barrier transmission of Dirac-like pseudospin-one particles *Phys. Rev.* B **84** 115136

[635] Xu H-Y and Lai Y-C 2016 Revival resonant scattering, perfect caustics, and isotropic transport of pseudospin-1 particles *Phys. Rev.* B **94** 165405

[636] Cheianov V V, Fal'ko V and Altshuler B L 2007 The focusing of electron flow and a Veselago lens in graphene p–n junctions *Science* **315** 1252–5

[637] Lee G-H, Park G-H and Lee H-J 2015 Observation of negative refraction of Dirac fermions in graphene *Nat. Phys.* **11** 925–9 letter

[638] Chen S *et al* 2016 Electron optics with p–n junctions in ballistic graphene *Science* **353** 1522–5

[639] Xu H-Y and Lai Y-C 2019 Pseudospin-1 wave scattering that defies chaos Q-spoiling and Klein tunneling *Phys. Rev.* B **99** 235403

[640] Akis R, Ferry D K and Bird J P 1997 Wave function scarring effects in open stadium shaped quantum dots *Phys. Rev. Lett.* **79** 123–6

[641] Akis R, Bird J P and Ferry D K 2002 The persistence of eigenstates in open quantum dots *Appl. Phys. Lett.* **81** 129–31

[642] Harayama T, Davis P and Ikeda K S 2003 Stable oscillations of a spatially chaotic wave function in a microstadium laser *Phys. Rev. Lett.* **90** 063901

[643] Lee S-Y, Kurdoglyan M S, Rim S and Kim C-M 2004 Resonance patterns in a stadium-shaped microcavity *Phys. Rev.* A **70** 023809

[644] Fang W, Yamilov A and Cao H 2005 Analysis of high-quality modes in open chaotic microcavities *Phys. Rev.* A **72** 023815

[645] Lebental M, Lauret J S, Hierle R and Zyss J 2006 Highly directional stadium-shaped polymer microlasers *Appl. Phys. Lett.* **88** 031108

[646] Nöckel J U, Stone A D and Chang R K 1994 Q spoiling and directionality in deformed ring cavities *Opt. Lett.* **19** 1693–5

[647] Mekis A, Nöckel J U, Chen G, Stone A D and Chang R K 1995 Ray chaos and Q spoiling in lasing droplets *Phys. Rev. Lett.* **75** 2682–5

[648] Nöckel J U and Stone A D 1997 Ray and wave chaos in asymmetric resonant optical cavities *Nature* **385** 45–7

[649] Xu H-Y, Wang G-L, Huang L and Lai Y-C 2018 Chaos in Dirac electron optics: emergence of a relativistic quantum chimera *Phys. Rev. Lett.* **120** 124101

[650] Ramos J G G S, da Silva I M L and Barbosa A L R 2014 Anomalous entanglement in chaotic Dirac billiards *Phys. Rev.* B **90** 245107

[651] Yusupov J, Otajanov D, Eshniyazov V and Matrasulov D 2018 Classical and quantum dynamics of a kicked relativistic particle in a box *Phys. Lett.* A **382** 633–8

[652] Ihnatsenka S and Kirczenow G 2012 Effect of electron–electron interactions on the electronic structure and conductance of graphene nanoconstrictions *Phys. Rev.* B **86** 075448

[653] Xu H and Lai Y-C 2017 Superscattering of a pseudospin-1 wave in a photonic lattice *Phys. Rev.* A **95** 012119

[654] Wang C-Z, Xu H-Y, Huang L and Lai Y-C 2017 Nonequilibrium transport in the pseudospin-1 Dirac-Weyl system *Phys. Rev.* B **96** 115440